GRAPHICAL SOLUTION OF THE TORSION PROBLEM

BY

M. I. I. RASHED,

B. Eng., Dr. Sc. tech.

The following is a trial to solve the torsion problem graphically. We begin with the fundamental strain equations, proceed to the assumptions of Saint Venant for the twisting of a prism, reform the corresponding equations until we reach a differential equation which can be solved graphically. The boundary conditions are introduced through a supplementary equation which is of the integral type.

STRESS AND STRAIN IN A TWISTED PRISM

(a) *General strain conditions in a body:*

We assume a body which is to deform under the action of external forces. Any body point is to have displacement components in the axis-directions, *i.e.* the point which initially has the cordinates (x, y, z) will have after the action of external forces the coordinates given by (x+u, y + v, z + w), where u, v and w represent the displacement components. They are functions of the coordinates x, y and z and they possess the following derivatives

$$\begin{array}{ccccc} \dfrac{\partial u}{\partial x} & , & \dfrac{\partial u}{\partial y} & , & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & , & \dfrac{\partial v}{\partial y} & , & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & , & \dfrac{\partial w}{\partial y} & , & \dfrac{\partial w}{\partial z} \end{array}$$

These represent the rate of change of displacement components with respect to the three independant variables x, y and z.

Assuming now another point near the point (x, y, z) having the coordinates $(x+\xi,\ y+\eta,\ z+\xi)$, then this new point will be displaced to a new point having the coordinates given by

$$x + u + \xi.\frac{\partial u}{\partial x} + \eta.\frac{\partial u}{\partial y} + \xi.\frac{\partial u}{\partial z}$$

$$y + v + \xi.\frac{\partial v}{\partial x} + \eta.\frac{\partial v}{\partial y} + \xi.\frac{\partial v}{\partial z}$$

$$z + w + \xi.\frac{\partial w}{\partial x} + \eta.\frac{\partial w}{\partial y} + \xi.\frac{\partial w}{\partial z}$$

In figure 1 is represented the displacement of a small cube.

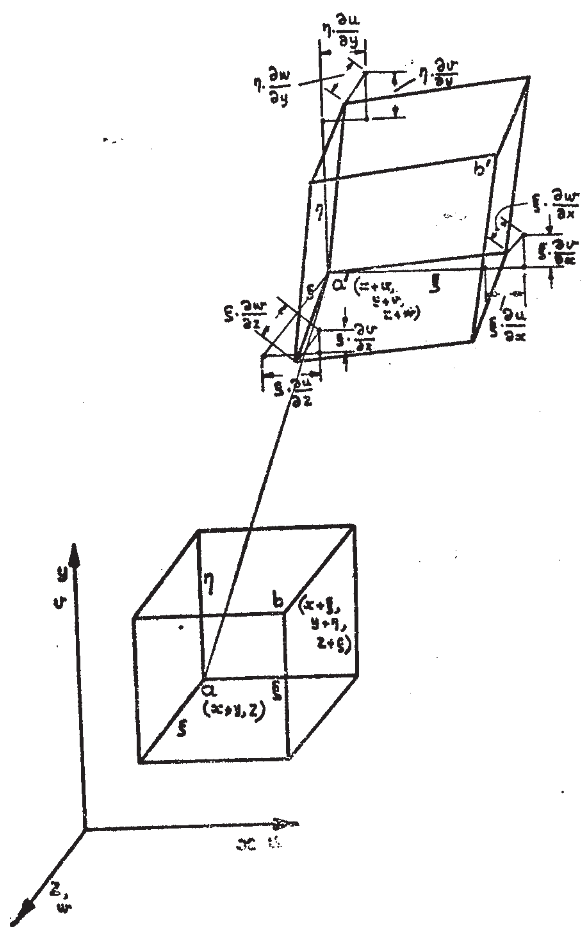

Fig. 1

To clarify the deformation due to the displacement in the three dimensional case we give in figure 2 the case of two dimensional displacement. In this case we have a small rectangular element a' bcd with the sides parallel to the x- and y- axis.

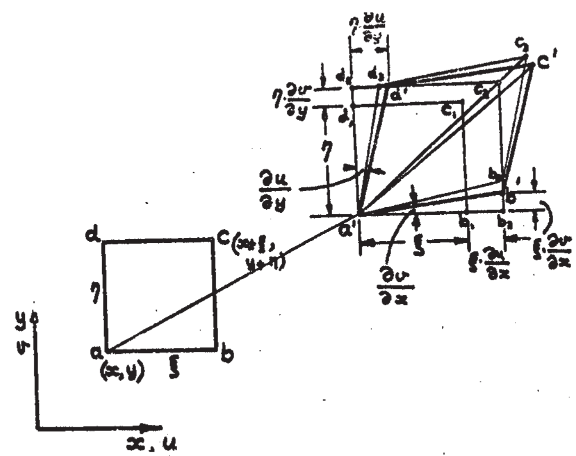

Fig. 2

After displacement the rectangular element will take the form of a parallelogram a'b'c'd'. The deformation of the rectangular element can be divided into:

(1) Elongation in the x and y directions. The corresponding strains in these directions are given by $\partial u/\partial x$ and $\partial v/\partial y$. If these strains occur alone, the rectangle will take the rectangular form $a'b_2c_2d_2$.

(2) Shear strain. The angle between the sides α b and α d is decreased by the value $\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ which represents the shear strain. If this deformation is added to the deformation mentioned in 1 we get the parallelogram a' b_3 c_3 d_3.

(3) Still we have to rotate the parallelogram by an angle c_3 a' c' to reach the final form α' b' c' d'. This angle of rotation is equal to $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Generalising now for a three dimensional case we get linear strains, shear strain components as well as rotational components. There are given by :

1. Linear strain components

$$e_{xx} = \frac{\partial u}{\partial x} \quad , \quad e_{yy} = \frac{\partial v}{\partial y} \quad , \quad e_{zz} = \frac{\partial w}{\partial z}$$

2. The shear strain components

In the x-y plane $\quad e_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}$

In the z-y plane $\quad e_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}$

In the z-x plane $\quad e_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}$

3. The rotational components

In the x-y plane $\quad 2\,\omega_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

In the y-z plane $\quad 2\,\omega_x = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}$

In the z-x plane $\quad 2\,\omega_y = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}$

(b) *Stress conditions in a body:*

We assume a small cube having the surfaces parallel to the planes y-z, z-x and x-y. Such a cube is shown in figure 3. On

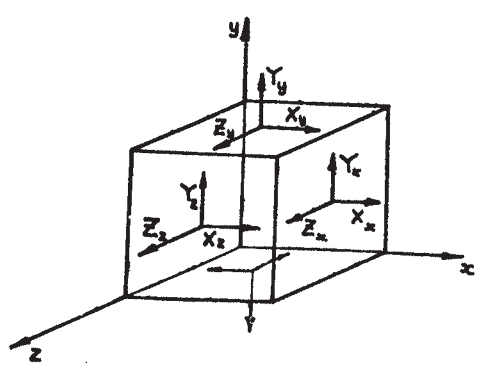

Fig. 3

a z- plane (a plane parallel to the x-y-plane), we have the components of the stress parallel to the x-, y- and z -axis given by X_z, Y_z and Z_z. The positive direction of the normal compoment is the tension direction or the outwards directed normal. If this direction coincides with the positive direction of one of the axes then we shall consider the positive directions of the other components parallel to the positive directions of the other axes. And if the positive direction of the normal component coincides with the negative direction of one of the axes, then the positive directions of the other components coincide with the negative directions of the other axes. It is easy to prove that:

$$X_y = Y_x \quad , \quad Y_z = Z_y \quad \text{and} \quad Z_x = X_z$$

And if there is no body forces then for equilibrium we have

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = 0$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0$$

(c) *Relation between strain and stress conditions :*

According to Hooke's law we have a *linear* relation between the strain and the stress. Hence we can apply the law of superposition. Thus we have the following relations valid

$$e_{xx} = \frac{1}{E} \cdot \left\{ X_x - \mu \left(Y_y + Z_z \right) \right\}$$

$$e_{yy} = \frac{1}{E} \cdot \left\{ Y_y - \mu \left(Z_z + X_x \right) \right\}$$

$$e_{zz} = \frac{1}{E} \cdot \left\{ Z_z - \mu \left(X_x + Y_y \right) \right\}$$

$$e_{xy} = \frac{1}{G} \cdot \quad X_x \quad , \qquad e_{yz} = \frac{1}{G} \cdot \; Y_z$$

$$e_{zx} = \frac{1}{G} \cdot \quad Z_x$$

where we have

E = Young's modulus of elasticity

μ = Poisson's ratio, the ration of the lateral contraction to the longitudinal extension.

G = The modulus of Rigidity.

(d) *Prism subjected to torsion:*

Theory of Saint Venant.

If we assume the cross-section of the prism to be circular, then according to the assumptions mentioned in elementary text books on theory of elasticity, we consider any normal cross-section to be turned relatively to any other cross-section through an angle proportional to the distance between the two planes. The shearing stress at any point is proportional to the distance of the point from the axis. The moment of the total shearing forces is equal to the twisting moment.

If the cross-section is not circular this assumption is not sufficient. We assume according to the theory of Saint Venant that the shear ***strain*** consists of two parts:

(*a*) A relative sliding in the traverse direction of elements of different cross-sections. This is the only type of strain occurring in a circular section.

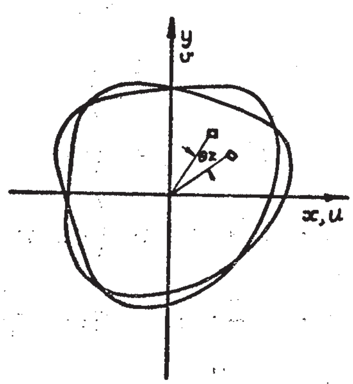

Fig. 4

(*b*) A relative sliding parallel to the length of the prism of different longitudinal linear elements. Due to this shear strain the plane cross-section becomes distorted into curved surface.

Taking the generating line of the surface of the prism parallel to the z-axis, we have for the displacement according to the shear strain mentioned under (a) the following relation:

$$u = -\theta . z . y \quad \text{and} \quad v = \theta . z . x$$

And we have for the displacement according to the shear strain mentioned under (b) the following relation:

$$w = \theta . \phi (x, y)$$

where

u, v and w represent the displacement components in the x, y and z directions.

θ = The twist per unit length.

Working the consequences of these assumptions we get for the strain components the following relations:

(1) Linear strains

$$e_{xx} = e_{yy} = e_{zz} = 0$$

(2) Shear strains

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta \left(\frac{\partial \phi}{\partial y} + x \right)$$

$$e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\theta \left(y - \frac{\partial \phi}{\partial x} \right)$$

(3) We still have for the rotational components

$$w_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \theta . z$$

$$w_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \frac{\theta}{2} \left(\frac{\partial \phi}{\partial y} - x \right)$$

$$w_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = -\frac{\theta}{2} \left(y + \frac{\partial \phi}{\partial x} \right)$$

According to the relations between the strain and stress mentioned on page 371, we get the following stress components that do not vanish :

$$Z_y = Y_z = G.\, e_{yz} = G.\, \theta. \left(\frac{\partial \varphi}{\partial y} + x \right)$$

$$Z_x = X_z = G.\, e_{zx} = G.\, \theta. \left(\frac{\partial \varphi}{\partial x} - y \right)$$

Writing the values of Z_x and Z_y in the equilibrium relation

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = o$$

already mentioned on page 371. we get

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = o =$$

$$= \frac{\partial}{\partial x} \left(G.\, \theta.\, \left(\frac{\partial \varphi}{\partial x} - y\right)\right) + \frac{\partial}{\partial y}\left(G.\, \theta.\, \left(\frac{\partial \varphi}{\partial y} + x \right)\right) + o$$

$$\text{or } \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = o$$

And this equation holds for all points of the cross-section. The function φ is so chosen as to satisfy this relation as well as the boundary conditions. With respect to the boundary conditions it is to be mentioned that the stress at any point on the boundary of the cross-section must be tangent to the boundary. Mathematically speaking these conditions are sufficient to define the function φ.

e) *Lines of shearing stress or shear lines:*

We have just seen that there exists a function φ such that the equation $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = o$ is satisfied.

According to the rules of the theory of functions (Differential and integral calculus of complex variables), there exists another function ψ such that

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = - \frac{\partial \psi}{\partial x}$$

where

$$(\varphi + i\,\psi) = F\,(x + i\,y)$$

We can also see that ψ satisfies the relation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Introducing ψ in the shear strain equations we get

$$\partial_{yz} = \theta.\left(\frac{\partial\,\varphi}{\partial\,y} + x\right) = \theta\left(-\frac{\partial\,\psi}{\partial\,x} + x\right)$$

$$= \theta\left(-\frac{\partial\,\psi}{\partial\,y} + \tfrac{1}{2}\,\frac{\partial\,(x^2 + F\,(y))}{\partial\,x}\right)$$

$$\partial_{zx} = \theta.\left(\frac{\partial\,\varphi}{\partial\,x} - y\right) = \theta\left(\frac{\partial\,\psi}{\partial\,y} - y\right)$$

$$= \theta\left(\frac{\partial\,\psi}{\partial\,y} - \tfrac{1}{2}\,\frac{\partial\,(y^2 + F\,(x))}{\partial\,y}\right)$$

Writing

$$F\,(x) = x^2 \text{ and } F\,(y) = y^2$$

we get

$$e_{yz} = -\,\theta\,\frac{\partial}{\partial\,x}\left(\psi - \tfrac{1}{2}\,(x^2 + y^2)\right)$$

$$e_{zx} = \theta\,\frac{\partial}{\partial\,y}\left(\psi - \tfrac{1}{2}\,(x^2 + y^2)\right)$$

Hence we get for the shear strees components

$$Z_y = -\,G.\,\theta.\,\frac{\partial\,(\psi - \frac{1}{2}\,(x^2 + y^2))}{\partial\,x}$$

$$Z_x = G.\,\theta.\,\frac{\partial\,(\psi - \frac{1}{2}\,(x^2 + y^2))}{\partial\,y}$$

Writing

$$\psi - \tfrac{1}{2}\,(x^2 + y^2) = \psi'$$

Then the equations for the shear stress components can be written as follows:

$$Z_y = -\,G.\,\theta.\,\frac{\partial\,\psi'}{\partial\,x}$$

$$Z_x = G.\theta.\frac{\partial\psi'}{\partial y}$$

ψ' is so chosen that the shear stress or the shear strain at any point on the boundary is always tangent to the boundary. To discuss this let us calculate the shear stress components in any direction. Let x' be another direction making an angle α with the x-direction as shown in figure 5. In this direction the shear

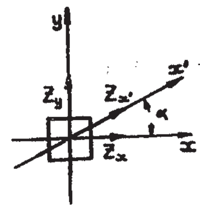

Fig. 5

stress component is given by

$$Z_{x'} = Z_x.\cos\alpha + Z_y.\sin\alpha$$

$$= G.\theta\left(\frac{\partial\psi'}{\partial y}\cos\alpha - \frac{\partial\psi'}{\partial x}\sin\alpha\right)$$

According to the calculus rules we have :

$$\delta\psi' = \frac{\partial\psi'}{\partial y'}.\delta y' + \frac{\partial\psi'}{\partial x'}.\delta x'$$

$$\therefore \quad \frac{\delta\psi'}{\partial y} = \frac{\partial\psi'}{\partial y'}.\frac{\partial y'}{\partial y} + \frac{\partial\psi'}{\partial x'}.\frac{\partial x'}{\partial y}.$$

$$\frac{\partial\psi'}{\partial x} = \frac{\partial\psi'}{\partial y'}.\frac{\partial y'}{\partial x} + \frac{\partial\psi'}{\partial x'}.\frac{\partial x'}{\partial x}$$

Thus we get

$$Z_{x'} = G.\theta\left(\frac{\partial\psi'}{\partial y'}.\frac{\partial y'}{\partial y} + \frac{\partial\psi'}{\partial x'}.\frac{\partial x'}{\partial y}\right).\cos\alpha$$

$$- G.\theta\left(\frac{\partial\psi'}{\partial y'}.\frac{\partial y'}{\partial x} + \frac{\partial\psi'}{\partial x'}.\frac{\partial x'}{\partial x}\right).\sin\alpha$$

But we have the following relations between x', y' and x, y.

$$x' = x.\cos\alpha + y.\sin\alpha$$

$$y' = y.\cos\alpha - x.\sin\alpha$$

The partial derivatives of x′ and y′ with respect to x and y are given by

$$\frac{\partial x'}{\partial x} = \cos\alpha \quad , \quad \frac{\partial x'}{\partial y} = \sin\alpha$$

$$\frac{\partial y'}{\partial y} = \cos\alpha \quad , \quad \frac{\partial y'}{\partial x} = -\sin\alpha$$

Introducing these in the above relation to get the value of $Z_{x'}$ we get

$$Z_{x'} = G.\theta \left\{ \left(\frac{\partial \psi'}{\partial y'} . \cos\alpha + \frac{\partial \psi'}{\partial x'} . \sin\alpha \right) \cos\alpha - \left(-\frac{\partial \psi'}{\partial y'} . \sin\alpha + \frac{\partial \psi'}{\partial x'} . \cos\alpha \right) \sin\alpha \right\}$$

or $$Z_{x'} = G.\theta . \frac{\partial \psi'}{\partial y'}$$

This means that the shear stress component in a certain direction is equal to $(G.\theta)$ multiplied by the partial derivative of ψ' in a direction leading this direction by 90°.

It is clear that to have the shear stress at a point on the boundary parallel to the boundary means that the shear component normal to the boundary equals zero. Hence the partial derivative of ψ' in the boundary direction at any point on the boundary must vanish. In other words ψ' takes a certain constant on the boundary. Let this constant value be zero.

We can now join all points of the cross-section at which ψ' takes a certain constant value. The resulting closed curve (or curves) represents a certain contour line.

According to the above reasoning the direction of the shear stress at any point is always tangent to the contour line at this point.

Let these contour lines be called shear lines or lines of shearing stress. The orthogonal lines we shall call the normal lines.

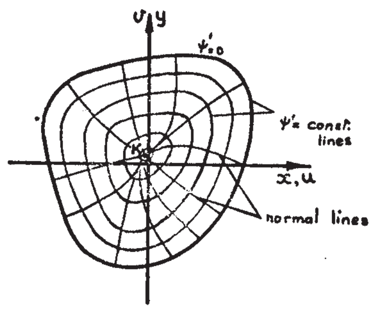

Fig. 6

Hence the shear stress at any point is equal to $(G.\theta)$ multiplied by the partial derivative of ψ' in the orthogonal line direction or in the normal line direction.

(f) *Calculation of the twisting moment:*

In figure 7 the element given by $\delta x.\ \delta y$ is subject to the

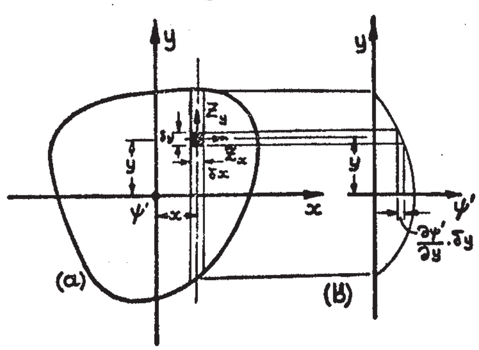

Fig. 7

shear stress components Z_x and Z_y. The moment of the shearing forces on this element about the z-axis is given by

$$\delta\ M_t = -\ Z_x\ .\ \delta\ x.\ \delta\ y.\ y + Z_y\ .\ \delta\ x.\ \delta y.\ x$$

$$= G.\theta\left(-\frac{\partial\psi'}{\partial y}\ .\ y.\ \delta\ x.\ \delta y\right.$$

$$\left.-\frac{\partial\psi'}{\partial y}\ .\ y.\ \delta x.\ \delta y\right)$$

Integrating we get

$$M_t = - G.\theta \left(\int\int y.\frac{\partial \psi'}{\partial y}.\ dx.\ dy + \int\int x.\frac{\partial \psi'}{\partial x}.\ dx.\ dy \right)$$

where the double integrals extend all over the cross-section.

With the aid of figure 7 let us try to get the value of the first integral. In figure 7 the coordinate system is a (x, y, ψ')-system. The x and y axes lie in the cross-sectional plane, the ψ' axis is perpendicular to both ψ' is a function of x and y and can be thus represented by a curved surface. This curved surface meets the x-y plane along the outer contour of the cross-section since along this contour we have taken $\psi' =$ zero.

In a plane parallel to both the ψ' and y axes (i. e. it is a x=constant plane) we have

$$\frac{\partial \psi'}{\partial y}.\ dy = d\ \psi'$$

Multiplying by y and integrating we get

$$\int y \frac{\partial \psi'}{\partial y}.\ dy = \int y.\ d\psi'$$

The value of this integral is represented by the hatched area A shown in figure 7b.

Integrating this area with respect to x we get that our double integral

$$\int\int y.\frac{\partial \psi'}{\partial y}.\ dy.\ dx = \int \left(\int y.\ d\psi' \right) dx = \int A.\ dx$$

represents the volume enclosed between the cross-section and the curved surface representing the relation between ψ' and x and y.

The second integral can also be proved to have exactly the same value as the first integral.

Thus the twisting moment is numerically equal to $G.\theta$ multiplied by twice the volume above mentioned. *i. e.* we have

$$M_t = -2\,G.\,\theta.\,V$$

$$\theta = \frac{-M_t}{2\,G.\,V}$$

where V represents the volume enclosed between the cross-section and the surface representing the relation between ψ' and x and y.

(g) *Graphical solution:*

Assume ψ' to represent a certain displacement function such that

$$u_x = \frac{\partial\,\psi'}{\partial\,y} \text{ and } u_y = -\frac{\partial\,\psi'}{\partial\,x}$$

where $u_x =$ and u_y represent the displacement components in the x and y directions. In any other direction x' the displacement component can be proved (*) to be given by

$$\text{Displacement component } u_{x'} = \frac{\partial\,\psi'}{\partial\,y'}$$

where y' represents the direction leading the x'-direction by 90°.

Thus along a normal line there is no displacement whatsoever.

The only displacement that exists is in the direction of the shear lines or lines of shearing stress (ψ'=constant).

Of the characteristic invariants of the function ψ' is the rotational component in the z -direction. By the rotational vactor component in the z -direction we mean

$$2.\ \text{Rotational vector component} = \frac{\partial v_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial\psi'}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\psi'}{\partial y}\right)$$

where x and y- axes are any two perpendicular to each other axes.

(*) The proof follows the same lines mentioned on pages 376, 377.

We have but

$$\psi' = \psi - \tfrac{1}{2}(x^2 + y^2)$$

thus we get

$$2.\ \text{Rotational vector component} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x} + x\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y} - y\right)$$

$$= -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + 2$$

Referring to page 375. we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Hence

$$\text{Rotational vector component} = 1$$

Referring to the meaning of the rotational vector on page 370, we see that it is represented by half the summation of the angles of rotation of two small segments originally perpendicular to each other.

Taking two small segments PA and PB as shown in figure 8 where

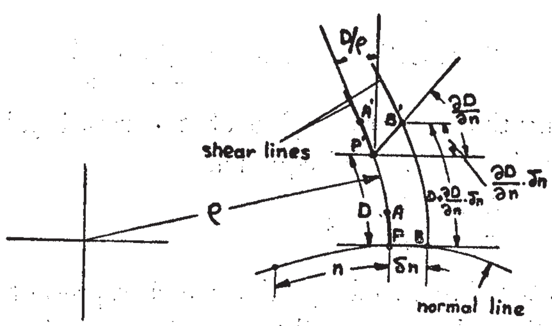

Fig. 8

—P is a point on a shear line and a normal line.

—PA is a segment of a small straight line on the shear line.

—PB is a segment or a small straight line on the normal line.

At the point P we have

ρ = The radius of curvature of the shear line.

n = The length of the normal line from a starting point on it to the point P.

n = The length of the segment PB.

The displacement of the point P is given by D and is equal to the length of the segment of the shear line from P to P'.

Then the displacement of the point B to the point B' is thus given by

$$D + \frac{\partial D}{\partial n} . \delta n$$

From the geometry of figure 8 we have

—The rotation of the segment PA to P'A' is through an angle equal to D/ρ.

—The rotation of the segment PB to P'B' is through an angle given by

$$\frac{\frac{\partial D}{\partial n} . \delta n}{\delta n} = \frac{\partial D}{\partial n}$$

Hence

$$2 . \text{ Rotational vector} = \frac{\partial D}{\partial n} + \frac{D}{\rho}$$

Equating this with the value of the rotational vector, we get the following differential equation

$$\frac{\partial D}{\partial n} + \frac{D}{\rho} = 2$$

This is the differential equation which represents the fundamental equation which we shall try to solve graphically. If for a certain cross-section the shapes of the shear lines are known, then the radius of curvature at any point is known and consequently the differential equation can be solved. The solution is found graphically by the method of trial and error, thus getting the value of D at any point

However if $D = o$ and $\rho = o$ [illegible]ed detailed discussion to get the value of D/ρ. This is the case when the shear line $\psi' =$ constant is one single point. Very near to this point the traction lines are represented by ellipses. For one of these ellipses assume the lengths of the major and minor axes to be equal to a and b. On this ellipse for the end points on the major axis the radius of curvature is equal to b^2/a and for the end points on the minor axis the radius of curvature is equal to a^2/b.

From the calculus rules we have

$$\frac{D}{\rho} \simeq \frac{\partial D}{\partial \rho}$$

when both D and ρ tend to zero.

For the point on the major axis we have

$$\delta n = \alpha$$

$$\delta \rho = \frac{b^2}{\alpha}$$

Hence it follows

$$\frac{D}{\rho} \simeq \frac{\partial D}{\partial \rho} \simeq \frac{\partial D}{\partial n} \cdot \frac{\partial n}{\partial \rho} = \frac{\partial D}{\partial n} \cdot \frac{\delta n}{\delta \rho}$$

$$= \frac{\partial D}{\partial n} \cdot \frac{a^2}{b^2}$$

Introducing the value of D/ρ in the fundamental equation we get

$$\frac{\partial D}{\partial n} + \frac{D}{\rho} = \frac{\partial D}{\partial n} + \frac{\partial D}{\partial n} \cdot \frac{a^2}{b^2} = 2$$

or

$$\frac{\partial D}{\partial n} = \frac{2}{(1 + a^2/b^2)}$$

In a similar way for the point on the minor axis it can be proved that

$$\frac{\partial D}{\partial n} = \frac{2}{(1 + b^2/a^2)}$$

With these equations it is possible to get the value of $\partial D | \partial n$ at the point where both D and ρ take the zero value.

This point which we shall call the *neutral point* represents the starting point of the integration of the fundamental differential equation.

$$\frac{\partial D}{\partial n} + \frac{D}{\rho} = 2$$

The integration follows a normal line.

(h) *Supplementary equation:*

We add to the above fundamental differential equation a supplementary equation having the integral type to add in the solution of our problem. This supplementary equation is now discussed.

In a two dimensional plane, if U and V are finite functions in x and y, single valued and differentiable at all points of a simply connected region which is completely bounded by a closed curve S, then GREEN's theorm states the following

$$\int \left(l.\ U + m.\ V \right) ds = \int\int \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} \right) dx.\, dy$$

where l and m represent the direction cosines of the inward directed normal to the curve S. The first integral is the line integral along the curve S in the anticlockwise direction, while the second integral extends all over the bounded region.

Remembering that the stress components Z_x and Z_y satisfy the above conditions for U and V, we can thus introduce

$$Z_x = U \quad \text{and} \quad Z_y = V$$

in the above equation. Hence

$$\int \left(l.\ Z_x + m.\ Z_y \right) ds = \int\int \left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} \right) dx.\, dy$$

Referring to the equilibrium equation mentioned on page 371 and namely

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = o$$

and knowing that Z_z = zero, we see that the right-hand integral vanishes, thus simplifying the equation to

$$\oint \left(l. Z_x + m. Z_y \right) \cdot ds = o$$

$(l. Z_x + m. Z_y)$ is nothing but the shear stress component in the direction having l nad m its direction cosines. Denoting this shear stress component by Z_n we can thus write

$$\int Z_n . ds = o$$

Further consider the region enclosed by two normal lines and the boundary of the cross-section as shown in figure 9. The con-

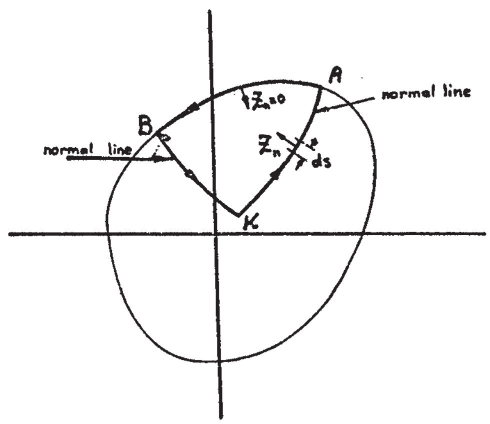

Fig. 9

tour of this region consists of the normal line KA, the part of the cross-section boundary AB and the normal line BK. The point K represents the neutral point, *i.e.* the point where the shear stress vanishes.

Along the part of contour AB we have ψ' = constant which means that the shear stress component normal to it is zero.

Along the normal lines KA and BK we have the normal stress component given by $(G. \theta)$ multiplied by the partial derivative of ψ' in the direction leading the stress component direction by 90°. For the normal line KA we thus get that the shear stress component in the direction of the inward directed normal is given by

$$Z_n = - G.\ \theta.\ \frac{\partial \psi'}{\partial n}$$

For the normal line BK the shear stress component in the direction of the inward directed normal is given by

$$Z_n = G.\ \theta.\ \frac{\partial \psi'}{\partial n}$$

where n represents the length of the corresponding normal line measured from a certain starting point which is taken in our case to be the neutral point K.

Introducing these in the line integral mentioned on the last page, we get

$$\int_K^A - G.\ \theta.\ \frac{\partial \psi'}{\partial n}.\ ds + \int_A^B o.\ ds + \int_B^K G.\ \theta.\ \frac{\partial \psi'}{\partial n}.\ ds = o$$

From K to A we have ds = dn and from B to K we have ds = — dn, thus getting

$$\int_K^A - G.\ \theta.\ \frac{\partial \psi'}{\partial n}.\ dn - \int_B^K G.\ \theta.\ \frac{\partial \psi'}{\partial n}.\ dn = o$$

Referring to the definition of the displacement (*) due to the displacement function ψ' we get that the value of $\frac{\partial \psi'}{\partial n}$ at any point is the displacement D.

Introducing D in the above equation we get

$$G.\ \theta.\ \int_K^A D.\ dn + G.\ \theta.\ \int_B^K D.\ dn = o$$

$$\therefore \quad \int_K^A D.\ dn = - \int_B^K D.\ dn$$

$$= \int_K^B D.\ dn$$

(*) The displacement is any direction equals the partial derivative of ψ' in a direction leading the displacement direction by 90°. The absolute value of the displacement at any point equals the partial derivative of ψ' in the normal line direction.

Since the choice of the normal lines K A and K B was quite arbitrary, we conclude that

$$\int D.\, dn = \text{constant}$$

where the line integral extends from the neutral point K along any normal line to the boundary of the cross-section.

This integral equation together with the fundamental equation

$$\frac{\partial D}{\partial n} + \frac{D}{\rho} = 2$$

solve the problem completely.

(i) *Check equation:*

In connection with the above-mentioned GREEN's theorm taking

$$U = -Z_y \text{ and } V = Z_x$$

we get

$$\int (-l.\, Z_y + m.\, Z_x).\, ds$$

$$= \int\int \left(\frac{\partial(-Z_y)}{\partial x} + \frac{\partial(Z_x)}{\partial y} \right) dx.\, dy.$$

As before the integral on the right-hand side extends all over a simply connected region and the integral on the left-hand side is the line integral in the anticlockwise direction along the boundary of the simply connected region. Refering to the values of Z_x and Z_y (page 375), introducing them in the above relation, *i.e.*

$$Z_x = G.\, \theta.\, \frac{\partial \psi'}{\partial y} \quad \text{and} \quad Z_y = -G.\, \theta.\, \frac{\partial \psi'}{\partial x}$$

wa get

$$\int (-l\, Z_y + m\, Z_x).\, ds = \int \left(+ l.\, G.\, \theta.\, \frac{\partial \psi'}{\partial x} + m.\, G.\, \theta.\, \frac{\partial \psi'}{\partial y} \right).\, ds$$

$$= G.\ \theta.\ \iint\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right).\ dx.\ dy$$

But $\quad \psi' = \psi - \frac{1}{2}(x^2 + y^2)$

where $\quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

hence $\quad \frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} = -2$

Finally we get

$$\int(-l.\ Z_y + m.\ Z_x).\ ds = -2\ G.\ \theta.\ A$$

where A represents the area of the simply connected bounded region.

We proceed now to discues the meaning of $(-l.\ Z_y + m.\ Z_x)$. l and m represent the direction cosines of the inwards normal to the boundary line of the simply connected region. This means that the direction cosines of the tangent to the curve or to ds are equal to m and $-l$. Referring to the ralation on page 376 between the stress components in any direction x' and the stress components in the x- and y-directions (*i.e.* Z_x and Z_y) which is given by

$$Z_{x'} = Z_x \cos\alpha + Z_y \sin\alpha$$

where $(\cos\alpha)$ and $(\sin\alpha)$ represent the direction cosines of the direction x'. Hence it follows

$$(-l,\ Z_y + m.\ Z_x) = Z_s$$

where Z_s represents the stress component in the S-direction. The stress component at any point in any direction is equal to $(G.\theta)$ multiplied by the displacement component in that direction due to the displacement function ψ'. Denoting this displacement component by D_s we get

$$\int(-l.\ Z_y + m.\ Z_x) = -2\ G.\ \theta\ A$$

$$\int Z_s.\ ds = \int G.\ \theta.\ D_s\ ds = -2\ G.\ \theta.\ A$$

$$\therefore \qquad \int D_s . \, ds = -2\,A$$

And this represtnts the check equation which will be used to check the results obtained from the fundamental equation together with the supplementary equation.

As a particular case we take the simply connected region to be the whole cross-section under discussion. If A_t denotes the area of the cross-section, D_b represents the displacement at a point on the boundary which is the absolute value of the displacement, then

$$\int D_b . \, ds = -2\,A_t$$

(j) ***Summary of the results:***

ψ' represents a certain displacement which satisfies the partial differential equation.

$$\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} = 2$$

In the cross-section the curves representing (ψ'=const.) are called shear lines or lines of shearing stress. The orthogonals to these lines are called the normal lines. The point where all the normal lines meet (ψ' is a maximum or a minimum) is called the neutral point K.

Due to the displacement function ψ' the displacement at any point in any direction is given by the partial derivative of ψ' in a direction leading the component direction by 90°. The displacement in the direction s is called D .

Along any normal line the displacement D satisfies the differential equation.

$$\frac{\partial D}{\partial n} + \frac{D}{\rho} = 2$$

where n denotes the length of the normal line from the neutral point K, ρ equals the radius of curvature of the shear lines at the intersection with the normal line.

For the integral way beginning from the neutral point K and ending on the boundary of the cross-section following any normal line, the following integral equation must be satisfied.

$$\int_{K}^{\text{Boundary}} D.\, dn = \text{Constant}$$

For the whole cross-section contour the line integral

$$-\int D_b.\, ds = 2\, A_t$$

supplies another relation. A_r= cross sectional area.

The stress at any point is given by (G.θ.D) where D denotes the displacement at that point, G equals the modulus of rigidity and θ the angle of twist per unit prism length.

The twisting moment is given by (-2G.θ.V) where V equals the volume enclosed between the (x-y) plane and the surface representing the relation $\psi' = F\ (x, y)$ in a (x, y, ψ')—space.

Practical steps of the solution:

(*a*) Assume a likely pattern of the shear lines and to the shear lines draw the corresponding normal lines. In many practical cases the shapes of the shear lines can be assumed with a reasonable degree of accuracy. Some approximate forms of these lines for various cross-sections are shown in figures 10 to 17.

(*b*) Choose a normal line I, use the fundamental differential equation

$$\frac{\partial D}{\partial n} + \frac{D}{\rho} = 2$$

together with the discussions on pages 382 and 383 leading to the values of D/ρ at the neutral point K, to get the relation between the displacement D and the length n of the normal line measured from the neutral point K as shown by curve I in figure 18. The radius of curvature ρ can be directly measured from the assumed shear lines.

(*c*) Repeat the same procedure as under (*b*) for another normal line II to get the curve II in figure 18.

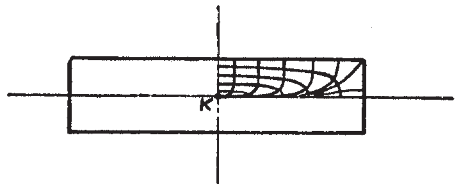

Fig. 10

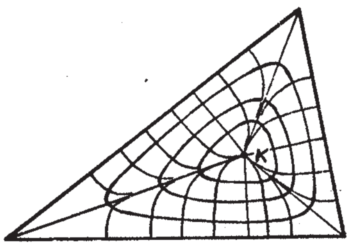

Fig. 11

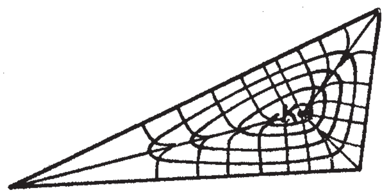

Fig. 12

Fig. 13

Fig. 14

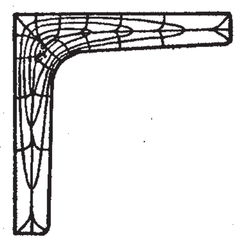

Fig. 15

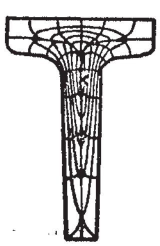

Fig. 16

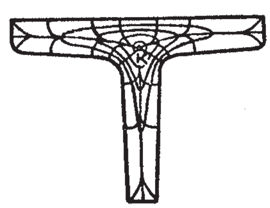

Fig. 17

(*d*) It is still preferable to repeat the same procedure under (*b*) for a third normal line III to get the curve III as shown in figure 18.

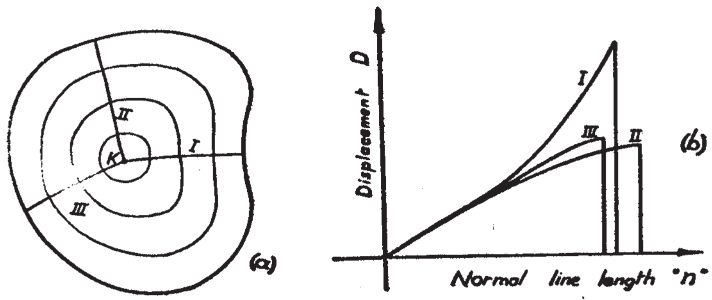

Fig. 18

(*e*) The supplementary equation.

$$\int_{K}^{\text{Boundary}} D.\, dn = \text{Constant}$$

states that the areas under curves I, II and III must be equal. If this is not the case, then it can be easily predicted in which direction the point K has to be shifted so that the corresponding areas become equal. In the case shown in figure 18 the normal line I has to decrease in length, the normal line III has to be increased in length, while the normal line II may be increased or decreased according to the new conditions. Repeat until the position of K is known.

(*f*) Having fixed the position of the neutral point K, repeat the procedure mentioned under (*b*) for a reasonable number of normal lines until the whole displacement field and subsquently the whole stress field are determined.

(*g*) Check the whole procedure using the check equation

$$\int D.\, ds = -2\, A_t$$

the path of integration being the boundary of the cross-section in the anticlockwise direction, A_t denotes the area of the section.

(h) Use the displacement curves to draw corrected pattern of the shear lines. Note that on a normal line we have

$$\frac{\partial \psi'}{\partial n} = D$$

(i) The procedure under (b) can now be repeated for a new normal line to convince the accuracy of the results.

(j) Referring to the discussions on pages 378, 379 and 380 about the twisting moment we see that

$$M_t = -2\, G.\, \theta.\, V$$

M_r denotes the twisting moment, V equals the volume enclosed between the cross-section and the surface representing the relation between ψ' and x and y in the (x, y, ψ')-space. This volume is given by

$$\int\int\int dx.\, dy.\, d\psi'$$

Cut this volume by a plane parallel to the x-y-plane (ψ = constant) to get a shear line. The area enclosed by this shear line is equal to

$$\int\int dx.\, dy = A_{\psi'}$$

Hence the volume V is given by

$$V = \int\int\int dx.\, dy.\, d\psi' =$$
$$\int \left(\int\int dx.\, dy\right) d\psi' = \int A_{\psi'}\, d\psi'$$

This is best done using the shear lines drawn as stated in (h) on this page.

Numerical Example:

In order to illustrate the proposed graphical method a numerical example for the twisting of a channel cross-section is shown in the accompaning plate.

The section used is No. 5 of the German standards DIN 1926. Dimensions as well as results are shown in the plate.