A STUDY OF THE INPUT CURRENT TO TRANSMISSION LINES UNDER AN APPLIED E.M.F., EU (T) SINWT

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INTRODUCTION

This paper deals with the determination of input transient and steady current responses due to the application of a pure sine wave voltage on electric lines. The solution is based on the use of Laplace transformations. Three cases are considered.

The first part deals with a smooth lossless line of distributed constants of inductive and capacitive reactances (L-Cline). The second part is a smooth transmission line with distributed constants, resistances and copacities. The third part deals with a line composed of lumped resistances and capacities (R-C circuit). The solutions of the cases given here are systematic and clear and give accurate responses with no approximation. It is felt that the solutions presented are new since no record has been found of previous solutions using the Laplace transform.

GENERAL EXPLANATION OF THE METHODS USED

In solving the differential equations of electrical circuits, it is possible to express the quantities involved in terms of a secondary variable. In terms of this secondary variable, the problem can be solved algebraically. Then by transforming back to the original independent variable, the solution to the original differential equation is obtained. The transformation which makes these operations possible is called the Laplace transformation.

The Laplace transformations transform f (t), a function of t into F (s), a function of some new variable s, according to the equation.

$$\int_0^{\infty} f(t)\, e^{-st}\, dt = F(s) \quad . \quad . \quad . \quad . \quad (1)$$

For example, if

$$f(t) = e^{-at} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

then

$$\int_0^{\infty} e^{-at} e^{-st}\, dt = \int_0^{\infty} e^{-(a+s)t}\, dt = \left. \frac{e^{-(a+s)t}}{-(a+s)} \right]_0^{\infty}$$

$$= \frac{1}{a+s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore, $F(s) = \frac{1}{a+s}$. Thus, the Laplace transformation allows e^{-at} to be expressed as $\frac{1}{a+s}$. The quantities e^{-at} and $\frac{1}{a+s}$ are, therefore, called Laplace transforms. Specifically $\frac{1}{a+s}$ is called the Laplace transform of e^{-at}, while e^{-at} is called the inverse transform of $\frac{1}{a+s}$.

Equation (1) is called the direct Laplace transformation. The physical significance of t in equation (1) is determined by the problem at hand, which will be time in the cases under consideration. On the other hand, the physical significance of a is just the significance which we find for it because of its relation to t in equation (1).

As we shall use it, t is restricted to real values, but s will be allowed to take on complex values. However, the real part of s will be required to be large enough to make the integral in equation (1) absolutely convergent. In accordance with the foregoing, it will be convenient frequently to refer to t as the real variable and s as the complex variable in the Laplace transformation.

A fundamental formula which is used in the solution of one of the cases is expressed by the following theorems. If

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

and $f(t) = 0$ when $t < o$

then $$e^{-bs} F(s) = \int_0^\infty f(t-b) e^{-st}\, dt$$

provided that b is a positive constant.

The direct Laplace transformation and the preceding theorem are used in solving the case of a smooth line of inductive and capacitive reactances.

The inversion thaorem for the laplace transformation is an integral formula by which f (t) may by obtained from F (s). The general statement of the inversion theorem is as follows: If F (s) is defined by the Laplace transformation.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad . \quad . \quad . \quad . \quad (1)$$

then $$f(t) = \frac{1}{2\pi j} \int_{y-j}^{y+j} F(s)\, e^{st}\, ds \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) can be expressed also as

$$f(t) = \frac{1}{2\pi j} \oint F(s)\, e^{st}\, ds \quad . \quad . \quad . \quad (5)$$

Equation (1) and (4) and their equivalent equations (1) and (5) are called inversion theorems. Therefore, f (t) is the sum of the residues of the functiou F (s) e^{st} with respect to all its singular points.

Equation (5) itself is at explicit formula for calculating the inverse of the Laplace transformation, and it is, therefore, called the inverse transformation. It can be used for finding f (t) from F (s) when the given form of F (s) cannot be treated with the aid of available tables of transforms. In fact, pairs of the Laplace transforms, f (t) and F (s), to be used for compiling a table, can be obtained from the inverse transformation (5) just as readily as from the direct Laplace transformation.

PART I

Smooth Lossless Line

The line consists of distributed constants of inductive and capacitive reactanees. The resistance and leakage conductances are considered to be equal to zero. An applied e.m.f. of a pure sine wave form is applied through an internal impedance equal to the characteristic impedance of the line. The length of the line is a finite value and the line is short circuited at the receiving end.

Different methods and ways of solutions were thoroughly tnd carefully investigated. These investigations led to the use of the wave equation.

This wave equation gives the most favorable solution.

The general wave equation is

$$I(s) = \frac{E(s)}{Z_s(s) + Z_C(s)} \left[e^{-\lambda(s)x} + b_2 e^{-\lambda(s)(2d-x)} \right] \left[1 + b_1 b_2 e^{-2\lambda(s)d} + b_1^2 b_2^2 e^{-\lambda(s)d} + \ldots \right] \qquad (1)$$

$$= \frac{E(s)}{Z_s(s) + Z_C(s)} \left[e^{-\lambda(s)x} + b_2 e^{-\lambda(s)(2d-x)} + b_1 b_2 e^{-\lambda(s)(2d+x)} + b_1 b_2^2 e^{-\lambda(s)(4d-x)} \ldots \right] \qquad (2)$$

where I (s), E (s), Z_s (s) and Z_C (s) are the Leplace transformations, for the circuit at a distance X from the sending end of the current, the applied sine wave valtage, the internal impedance of the source and the characteristic impedance of the smooth line, where λ (s) is the propagation constant of the line expressed in the Laplace transformation. Therefore, in this case $\lambda(s) = s\sqrt{CL}$. internal impedance of the source and the characteristic impedance of the smooth line, where λ (s) is the propagation constant of the line expressed in the Laplace transformation. Therefore, in this case $\gamma(s) = \sqrt{LC}$.

L = inductive reactance per unit length of the line,

C = capacitive reactance per unit length of the line

L and *C* are conatant values.

$$b_1 = \frac{1-a_1}{1+a_1} \quad \text{and} \quad b_2 = \frac{1-a_2}{1+a_2} \quad a_1 = \frac{Z_s(s)}{Z_C(s)} \text{ and } a_2 = \frac{Z_C(s)}{Z_r(s)}$$

where Z_r (s) is equal to the receiving end impedance expressed in the Laplace transformation form.

The first term in equation (2) represents the current wave in a very long reflectionless line, the second term represents a wave which has travelled to the receiver at $X=d$ (in the finite line considered) and is reflected back to X, the third represents a wave which has travelled to $X=d$, back to $X=0$, and thence to X, and so on.

In the case under consideration, where the line is considered to be short circuited at the receiving end, the equation is written as follows :

$$I(s) = \frac{E(s)}{2Z(s)} e^{-\lambda(x)} + e^{-\lambda(2d+x)} \tag{3}$$

The result of this equation is

$$I(t) = \frac{1}{2}\sqrt{\frac{L}{C}} \sin \omega (t - \sqrt{LC}\, x) + \frac{1}{2}\sqrt{\frac{C}{L}} \sin \omega t - \sqrt{LC}\,(2d-x). \tag{4}$$

The equation (4) represents the current as a function of the time at any point a distance X from the sending end on the line. This solution is based on the application of the direct Laplace transformation, taken from the tables, with the use of the theorem mentioned before in Part I. To get the input current, x is set equal to zero. Therefore, the input current as a function of the time is equal to

$$I_o(t) = \frac{1}{2}\sqrt{\frac{C}{L}} \sin \omega t + \frac{1}{2}\sqrt{\frac{C}{L}} \sin \omega (t - 2d\sqrt{LC}) \quad (5)$$

where $\sqrt{\frac{L}{C}}$ is equal to the characteristic impedance of the line. The term $\omega \sqrt{LC}$ is the propagation constant of the line.

The significance of the two terms which make up equation (5) is as follows. From time equal to zero to the time when the wave returns to the sending end, the first term of the equation is used. This term is $\frac{1}{2}\sqrt{\frac{C}{L}} \sin \omega t$.

The time for a round trip is equal to twice the length of the line d divided by the velocity of propagation of the line.

$$\text{Time} = \frac{2d}{v} = 2d\sqrt{LC}$$

After the time becomes equal to ($2d\sqrt{LC}$), the two terms of equation (5) are considered.

To get the input current for an infinite line, we consider equation (2)

$$I_o(s) = \frac{E(s)}{Z_C(s)} e^{-\lambda x}$$

Therefore, the equation of the current at any point X for an infinite lossless line is equal to

$$I(t) = \frac{1}{2}\sqrt{\frac{C}{L}} \sin \omega (t - \sqrt{LC}x).$$

Also the input current as a function of time for an infinite lossless line can be written as

$$I_0\,(t) = \frac{1}{2}\sqrt{\frac{C}{L}}\sin\omega\, t.$$

PART II

As R-C Smooth Line

The line consists of distributed constants of resistances and capacitances. The inductances and the leakage conductances are considered to be equal to zero. An e.m.f. of a pure sine wave form is used. The line is of a finite length and short circuited at the receiving end. Therefore,

$R_0 = \sin\omega t$

R = resistance per unit length of the line.

C = capacitive reactance per unit length of the line.

d = the length of the line.

The total current expressed in the Laplace transformation is as follows:

$$I_0\;(s) = \frac{\omega\, s}{\omega^2 + s^2}\sqrt{\frac{C}{Rs}}\,\frac{\cosh d\sqrt{RCs}}{\sinh d\sqrt{RCs}} \quad . \quad . \qquad (1)$$

$$Z_c\;(s) = \sqrt{\frac{R}{Cs}},\ \text{and}\ \lambda\;(s) = \sqrt{RCs}.$$

To solve equation (1), two theorems are considered. One is the inversion theorem of the Laplace transformation. The other is the Borel's theorem.

If $\quad f_1\;(s) = \frac{\omega\, s}{\omega^2 + s_2}$, then $f_1\;(t) = \omega\cos\omega\, t.$

Also, if $f_2\;(s) = \sqrt{\frac{C}{Rs}}\,\frac{\cosh d\sqrt{RCs}}{\sinh d\sqrt{RCs}}$,

then $f_2\;(t) = \frac{1}{d\,R} + \frac{2}{d\,R}\sum_{m=1}^{\infty} e^{-\frac{m^2\pi^2}{d^2RC}t}$

By using Borel's theorem, the input current as a function of time can be expressed as :

$$I_0(t) = \omega \int_0^t \cos \omega (t-T) \sum_{m=1}^{m=\infty} \left[\frac{1}{dR} + \sum_{m=1}^{\infty} \frac{2}{dR} e^{-KT} \right] dT$$

where

$$K = \frac{m^2 \pi^2}{d^2 RC}$$

Solving the above equation gives the total input current.

$$I_0(t) = \sum_{m=1}^{\infty} \left[\frac{-e^{-KT}}{dR} \sin 2\phi + \frac{1}{dR} \sin \omega t + \frac{2}{dR} \cos \phi \right.$$

$$\left. \sin(\omega t + \phi) \right] \qquad (2)$$

where $\tan \phi = \frac{K}{\omega}$.

From the total solution of equation (2), two states exist; the transient state:

$$\sum_{m=1}^{m=\infty} \frac{-e^{-Kt}}{dR} \sin 2\phi$$

and the steady state terms :

$$\frac{1}{dR} \sin \omega t + \sum_{m=1}^{\infty} \frac{2}{dR} \cos \phi \sin(\omega t + \phi).$$

As a tentative check on equation (2) it is noted that $I_0(t)$ reduces to zero when $t=0$. Also, when t approaches infinity, $I_0(t)$ reduces to the steady state portion only of equation (2).

PART III

The Lumped (R-C) Line

The line consists of lumped resistances and capacitive reactances (R-C). An e.m.f. of pure sinusoidal shape is applied. The length of the line is a finite value and the line is short circuited at the receiving end. The general equation for the current expressed in the Laplace transformation is as follows:

$$I_r(s) = \frac{E(s)\sinh\theta\cosh(m-r)\theta + \frac{Z'(S)}{Z'(S)}\sinh(m-r)\theta}{Z(s)\sinh\theta\,(\sinh m\theta\sinh\theta + \frac{Z^H(S)}{Z(S)}\cosh m\theta)} \qquad (1)$$

where Z (s), Z″ (s), Z′(s) are impedances expressed in the Laplace transformation.

In the case under consideration, the input current expressed in the Laplace transformation can be written as:

$$I(s) = \frac{E_o(s)\sinh\theta\cosh m\theta}{Z(s)\sinh\theta\sinh m\theta\sinh\theta} = \frac{\omega Cs\cosh m\theta}{(s^2+\omega^2)\sinh m\theta\sinh\theta}$$

Therefore,

$$I_o(t) = \frac{1}{2\pi j}\oint \frac{\omega Cs\cosh m\theta\, e^{st}}{s^2+\omega^2)\sinh m\theta\sinh\theta}\,ds. \qquad (2)$$

Equation (2) can be solved by the inversion theorem and calculus of residues. The solution of this equation gives two responses. One is the steady state response and the other is the transient response.

The transient response is given by the following:

$$I_{o1}(t) = \frac{-4\omega C}{m\,(\omega^2R^2C^2+16)} e^{\frac{-4}{RC}t} - \frac{2\omega}{mR}\sum_{P=1}^{m-1}\frac{\alpha}{\omega^2+^{2S}}e^{-\alpha t} \qquad (3)$$

where $\alpha = \frac{2}{RC}\left(1-\cos\frac{P\pi}{m}\right)$.

The above response is due to the poles of the function $\sinh\theta\sinh m\theta$.

The steady state response is given by,

$$I_0(t) - \frac{\omega C}{\sqrt{L^2+F^2}}\sin(wt+\phi) \qquad (4)$$

where

$$E + jF = \frac{\sqrt{JRC\omega - \frac{R^2C^2\omega^2}{4}}}{\coth m\theta_1} \text{ and } E - jF = \frac{\sqrt{-JRC\omega - \frac{R^2C^2\omega^2}{4}}}{\coth m\theta_2}$$

$$\phi = \tan^{-1}\frac{E}{F}, \quad \sinh\theta = \sqrt{RCs + \frac{R^2C^2s^2}{4}}$$

$$\text{and } \theta_1 = \theta\Big]_{s=j\omega}, \quad \theta_2 = \theta\Big]_{s=-j\omega}$$

The above steady state term is due to the poles of the function $(s^2+\omega^2)$. Therefore, the total input current for the m sections and short circuited at the receiving end is.

$$I_o(t) = \frac{-4\omega C}{m(\omega^2R^2C^2+16)} e^{\frac{-4}{RC}t} - \frac{2\omega}{mR}\sum_{P=1}^{m-1}\frac{\alpha}{\omega^2+\alpha^2} e^{-\alpha t}$$

$$+ \frac{\omega C}{\sqrt{E^2+F^2}}\sin(\omega t+\phi) \quad . \quad . \quad . \quad (5)$$

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