ANALYSIS OF CONTINUOUS FRAMES[1]

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Introduction

Since the introduction of the continuous frame, with rigid joints, many methods have been advanced for its structural analysis. From time to time, new methods are introduced to help eliminate the tedious work usually involved in the analysis. None of the present methods, however, could claim to give a practical solution to all problems ; and it seems that search should continue to arrive at easier and more practical solutions for these structures. The work here presented is a contribution towards this end.

This work is divided into three parts. In Part 1 an approximate direct moment distribution method is introduced. The method is a modification of the original Cross method, in which repeated cycles of distribution and carry-over are eliminated. Solutions are, therefore, made easier and the work cut to a minimum.

In Part 2 a method of solving continuous arched frames of two or three vents is presented. As an example, a three-vent one-storey frame with semi-elliptical roofs is solved. The solution is based on "Column Analogy" and "Virtual Work", which when combined together produce an easy and clear method of analysing such frames.

From a thesis for the M.Sc. degree submitted to the Faculty of Engineering, Cairo University, 1953, by S. G. Soliman.

In Part 3 an experimental investigation on frames by models is presented. The object of experiments was to study the elastic behaviour of rectangular haunched frames, and to eatablish an appropriate method for choosing the members into which a frame is assumed to be divided. Out of the results obtained, the elastic behaviour of a frame may be fairly accurately represented if its beams are extended to the axes of the adjoining columns ; while columns are taken to extend to the axes of the beams with part of these columns at the connections, considered as having an infinite moment of inertia. The lengths of these parts are given by two rules—deduced from experiments, one for one-sided haunched columns and the other for double haunched ones.

PART 1

An Approximate Direct Method of Moment Distribution

In this method a new procedure is adopted in the well-known method of "moment distribution" for the structural analysis of continuous frames ; the fundamental principles of the original method are maintained, but the repeated cycles of distribution and carry-over are eliminated. Hence, this method does away with the most tedious part of the original method ; especially when several cases of loading, or when influence lines are considered in the analysis of frames.

The method is based on the following two concepts :—

(*a*) The moment transmitted to the far end of a prismatic member is 1/2 of the moment at the near end, if the far end is fixed. This well-known coefficient or carry-over factor "r" becomes zero when the far end of the member is simply supported. If the far end is connected to other members, the carry-over factor "C" instead of "r", will lie between 0·5 and zero according to its actual conditions of restraint.

(*b*) In distributing an unbalanced moment at a joint say "a", the distribution is made in proportion to "D" factors instead of "K"; Dab being the distribution factor of member "ab", with its far end "b" under actual conditions of restraint. This factor depends on the carry-over factor Cab.

General expressions for the carry-over factors "C", and the distribution factors "D" can be derived as follows:

Consider the frame shown in Fig 1. Apply a moment Dab at "a", and let the bending moment diagram be as shown. The slope angle Dab of the elastic line at "a", due to Dab will be unity.

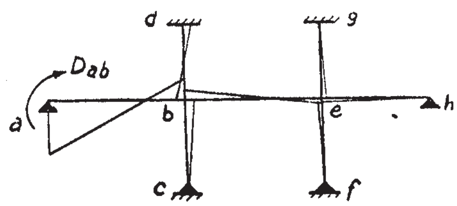


Fig. 1

$$\alpha\, ab = \frac{Dab\ L}{3\ E\ Iab} - \frac{Cab\ Dab\ L}{6\ E\ Iab} = 1$$

$$= \frac{Dab\ L}{3\ E\ Iab}\left(1 - \frac{Cab}{2}\right)$$

$$Dab = \frac{1{\cdot}5\ Kab}{2 - Cab} \quad \dots \quad \dots \quad \dots \quad \dots (1)$$

where $Kab = \frac{4\ E\ Iab}{Lab} =$ absolute value of stiffness of member ab with its end be considered as fixed.

$$\text{Now, } \alpha ba = \frac{Cab\ Dab\ L}{3\ E\ Iab} - \frac{Dab\ L}{6\ E\ Iab} = \frac{Dab\ L}{3\ E\ Iab}\left(Cab - \frac{1}{2}\right)$$

$$= \frac{4\ Dab}{3\ Kab}\left(Cab - \frac{1}{2}\right) \dots (2)$$

But $\alpha ba = \alpha bc = \alpha bd = \alpha be$

$$\alpha ba = \frac{Mbc}{Dbc} = \frac{Mbd}{Dbd} = \frac{Mbe}{Dbe} \dots (3)$$

and

$$-Cab\ Dab = Mbc + Mbd + Mbe \dots (4)$$

where Mbc, Mbd, and Mbe are the moments carried to ends "b" of members bc, bd and be.

From (3) and (4) we have :

$$—Cab\ Dab = \alpha ba\ (Dbc + Dbd + Dbe)$$
$$= \alpha ba\ \Sigma\ D \quad \ldots \quad \ldots \quad \ldots (5)$$

From (5) and (2) we have :

$$—Cab\ Dab = \frac{4\ Dab}{3\ Kab}\left(Cab - \frac{1}{2}\right)\Sigma\ D$$

Therefore :

$$Cab = \frac{2\ \Sigma\ D}{3\ Kab + 4\ \Sigma\ D} \quad \ldots (6)$$

Equations (1) and (6) give the required expressions for the actual distribution factor Dab at end "a" of member "ab", and the carry-over factor Cab from "a" to "b".

The term Σ D is the sum of the distribution factors of the ends of the members branching from joint "b", excluding member ab. If any of these members say "bd" has a fixed end, then, its distribution factor Dbd = 4 E Ibd/Ibd. If any of these members say "ac" is hinged at "c", then, its distribution factor will be 3E Ibc/Lbc. And if any of these members such as "be" is continuous over other joints, then, its distribution factor Dbe is proposed to be taken Dbe = 3·5 E Ibe/Lbe considering its restraint at its far end "e" as intermediate between a fixed ended and a hinged ended.

Since relative values only could be used in equations (1) and (6), the relative values of the distribution factors :

Dbd : Dbc : Dbe = Kbd : 3/4 Kbc : 7/8 Kbe.
in which K = I/L = relative stiffness for each member.

The accuracy of this approximation was tested by consideration of the continuous beam in Fig. 2, which can well be taken to represent the action of the frame in Fig. 1, since the flexural properties of any members branching from b and e could be accounted for by increasing the value of K_2 and K_3 in Fig. 2.

Fixing the values of K_1 at unity and assigning various values to $K_1 : K_2$ and $K_3 : K_2$ extending over wide ranges, the values of Cab and Dab were calculated by the proposed method as well as

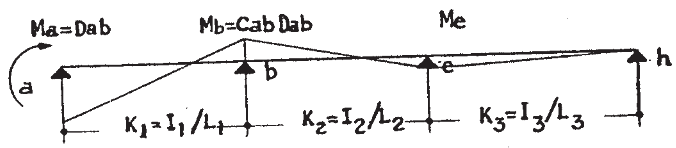


Fig. 2

by the equation of three moments. Comparing the results we arrive at the following two conclusions :—

1. The approximate values of Cab are very close to the corresponding exact values for normal relative proportions of $K_1 : K_2$ and $K_3 : K_2$. For abnormal values of these ratios, the difference seldom exceeds 5 %. These small differences occur only when the "C" values become very small, and in these cases the carried over moments will consequently be also very small, and the ultimate effect of these small differences on the final analysis will be negligible.

2. The approximate and the exact values of Dab are almost identical. This is an important feature since the distribution factor has, by far, the greater effect on the analysis of frames than the carry-over factor.

Description of Method :

The method is explained in the following steps :

For the shown frame Fig. 3. of prismatic members.

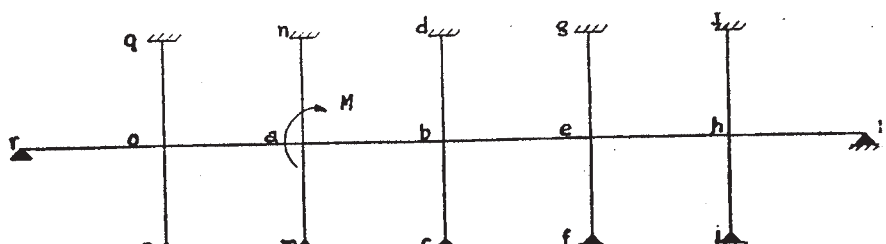


Fig. 3.

1. Calculate for each end of every member of the frame the C and D values according to equations (6) and (1). For example:

$$Cab = \frac{2\ \Sigma\ D}{3\ Kab + 4\ \Sigma\ D} = \frac{2\ (Kbd + 3/4\ Kbc + 7/8\ Kbe)}{3\ Kab + 4\ (Kbd + 3/4\ Kbc + 7/8\ Kb)}$$

$$Dab = \frac{1{\cdot}5\ Kab}{2 - Cab}$$

2. An unbalanced moment M at Joint "a" is first distributed in proportion to the values Dab, Dam, Dan and Dao of ends of members meeting at "a".

A moment "Mba" is carried over to end b of ab equals Cab times the distributed moment to end a. This carried moment is balanced by members bc, bd and be according to their D factors. No moment is carried to c since it is a hinged end. A moment Mbd is carried to the fixed end d equals to half Mbd. A moment Meb is carried to end e of span be, equals Cbe times Mbe.

Moment Meb carried to end e is then treated in a similar manner as that carried to end b of ab.

Moment distributed to end "a" of members an, am and ao of joint a are treated in the same way as that of ab.

3. Unbalanced moments at joints may either be taken one by one, or taken simultaneously; the former procedure will be found clearer and more rapid.

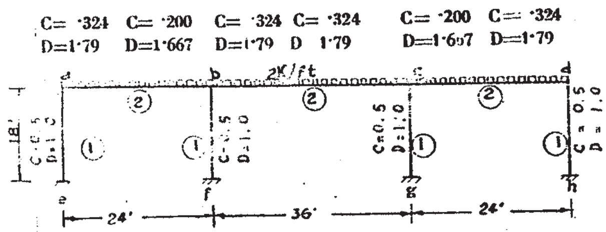
4. (*a*) In carrying over from an end of a member say a of ab, it will be found—in almost all cases—that the process of distribution and carry-over is seldom needed after two subsequent joints, *i.e.* the moment carried to "h" may be taken as zero.

(*b*) In many other cases, it is found that the process could be stopped after only one subsequent joint, *i.e.* the moment carried to joint "e" could be taken fairly accurately as zero.

These two procedures will usually suggest themselves, when the moment carried over is found to be too small to be distributed and carried any further.

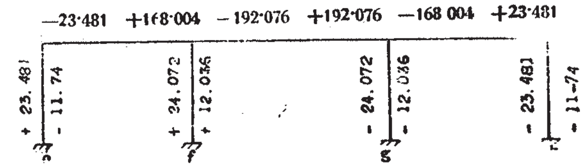
Numerical Example:

The method will now be illustrated by the numer[illegible] [illegible]ample in Fig 4. This shows a symmetrical frame of prismatic members, symmetrically loaded. The relative values of K= I/L for the members are indicated in circles. The final moments are compared to those found by the ordinary moment distribution method using 6 cycles.



F.E.M. . . .	−99·00	+96·000	−216·000	+216·000	−96·000	+96·000
Joint "a". . .	+61·591→	+19·955	− 12·803→	− 4·148	+ 2·[illegible]	0·0
Joint "b". . .	+ 8·976←	+44·882	+ 48·194→	+ 15·615	− 9·760→	− 1·[illegible]52
Joint "c". . .	+ 1·952←	+ 9·760	− 15·615←	− 48·194	−44·882→	− [illegible]·976
Joint "d". . .	0·0	− 2·593	+ 4·148←	+ 12.803	−19·955←	−17·591

Final Moments .	−23·481	+168·004	− 192·076	+192·076	−168 004	+23·481



Final Moments by ordinary mmt distribation method using 5 cycles.	−24·023	167·987	− 192·002	192·0[illegible]2	−167·987	+24·0[illegible]

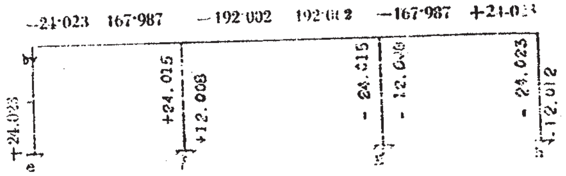


Fig. 4

1. The values of C and D are first obtained as follows:

Cab = 0·500

Dae = K = 1·00

$$Cab = \frac{2\ (6/8 \times 2{\cdot}0 + 1{\cdot}0)}{3 \times 2 + 4\ (7/8 \times 2 + 1{\cdot}0)} = 0{\cdot}324$$

$$\text{and } Dab = \frac{1{\cdot}5 \times 2{\cdot}0}{2 - 0{\cdot}324} = 1{\cdot}79$$

$$Cba = \frac{2 \times 1{\cdot}0}{3 \times 2 + 4 \times {\cdot}0} = 0{\cdot}200$$

$$\text{and } Dba = \frac{1{\cdot}5 \times 2{\cdot}0}{2 - 0{\cdot}20} = 1{\cdot}667$$

$$Cbc = \frac{2\ (7/8 \times 2 + 1{\cdot}0)}{3 \times 2 + 4\ (7/8 \times 2 + 1{\cdot}0)} = 0{\cdot}324$$

$$\text{and } Dbc = \frac{1{\cdot}5 \times 2{\cdot}0}{2 - 0{\cdot}324} = 1{\cdot}790$$

Because of symmetry calculations for joints d and c are taken from those for a and b.

2. The unbalanced moment at joint a "— 96" is distributed to ends "a" of members ab and ae in proportion to Dab and Dae giving:

$$Mab = \frac{1{\cdot}79}{(1{\cdot}79 + 1{\cdot}0)} \times 96 = +\ 61{\cdot}591 \text{ K. ft.}$$

A moment Mba $= 0{\cdot}324 \times 61{\cdot}591 = +\ 19{\cdot}955$ K. ft. is carried to end b of member ab. This "Mba" is balanced by moments Mbc and Mbf at ends "b" of members bc and bf in proportion to Dbc and Dbf giving:

$$Mbc - \frac{1{\cdot}79}{(1{\cdot}79 + 1{\cdot}00)} \times -\ 19{\cdot}955 = -\ 12{\cdot}803 \text{ K. ft.}$$

In turn, a moment Mcb $= 0{\cdot}324 \times -\ 12{\cdot}803 = -\ 4{\cdot}148$ K. ft. is carried to end "c" of member bc, and balanced by Mcd and Mcg in proportion to Dcd and Dcg giving:

$$Mcd = \frac{1{\cdot}667}{(1{\cdot}667 + 1{\cdot}0)}\ 4{\cdot}148 = +\ 2{\cdot}593 \text{ K. ft.}$$

No moment is to be carried to joint "d".

3. The unbalanced moment at joint b "120·00 K. ft." is distributed to ends b of members ba, bf and bc in proportion to their D values giving:

$$Mba = \frac{1{\cdot}667}{(1{\cdot}667 + 1{\cdot}0 + 1{\cdot}79)} \times 120 = + 44{\cdot}882$$

$$\text{and } Mbc = \frac{1{\cdot}79}{4{\cdot}457} \times 120 = + 48{\cdot}194$$

A moment Mab $= 0{\cdot}2 \times 44{\cdot}882 = + 8{\cdot}976$ is carried to end "a" of member ab and balanced by Mae.

A moment Mcb $= 0{\cdot}324 \times 48{\cdot}194 = + 15{\cdot}615$ is carried to end "c" of member cb, and balanced by Mcd and Mcg in proportion to Dcd and Dcg giving:

$$Mcd = \frac{1{\cdot}667}{(1{\cdot}667 + 1{\cdot}0)} \times (-15{\cdot}615) = -9{\cdot}760 \text{ K. ft.}$$

In turn a moment Mdc $= 0{\cdot}2 \times (-9{\cdot}76) = -1{\cdot}952$ K. ft is carried to end d of cd, and balanced by Mdh.

4. Because of symmetry, calculations for joint "d" are directly taken from those for joint "a", and those for "c" are taken from joint "b".

5. The final moments at ends of members are obtained by adding the fixed-end-moment, the distributed moment and the carried or balancing moments at each end.

Final moments at top ends of columns are found at the end of the solution from the final girder moments.

Conclusions:

The methed has been tested by several examples of different nature, *e.g.* continuous beams, frames, multi-storey frames, with and without joint translation (sway), secondary stresses (moments) in a loaded truss, and a vierendeel girder. The results obtained were extremely accurate.

Apart from the principal concepts of the original moment distribution method, only two formulae for D and C factors are added. Although these formulae are easy to remember and the factors easy to calculate, a set of tables are prepared from which these factors are directly obtainable.

When several cases of loading are studied, or when influence lines are required, the mothod is at an advantage and affords an easy solution for these cases.

PART 2

Extension to Column Analogy

This part deals with the structural analysis of single-storey portal frames, continuous over two or three vents. Methods of solving such frames are usually tedious. The method herein presented is probably one of the simplest available as it maintains the merits of the valuable method of "Column Analogy" and the simplest application of "Virtual Work".

Description of Method:

Consider the two-vent frame in Fig. 5 a. Using the terminology of "virtual work method"; we chose our main system as the fixed frame "ac", by cutting at "b" (Fig. 5-b).

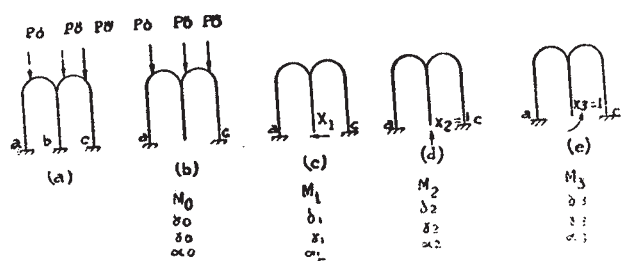


Fig. 5

The redundant elements will therefore be X_1, X_2, X_3 each acting once at "b".

Apply separately to the main system $X_1 = 1$, $X_2 = 1$ and $X_3 = 1$ (Fig. 4-c, d, e).

Let δ_o, γ_o, σ_o be the horizontal, the vertical and angular displacements at b of main system due to the given loads P_o.

Let δ_1, γ_1, α_1 be the displacements corresponding to $X_1 = 1$, and δ_2, γ_2, α_2 be the displacements corresponding to $X_2 = 1$, and δ_3, γ_3, α_3 be the displacements corresponding to $X_3 = 1$. The three redundants are obtainable from the conditions:

$$\left.\begin{aligned} \delta_o + X_1\,\delta_1 + X_2\,\delta_2 + X_3\,\delta_3 &= 0 \\ \gamma_o + X_1\,\gamma_1 + X_2\,\gamma_2 + X_3\,\gamma_3 &= 0 \\ \alpha_o + X_1\,\alpha_1 + X_2\,\alpha_2 + X_3\,\alpha_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (1)$$

If Mo, M1, M2 and M3 are the bending moment diagrams of the main system due to Po, X1, X2 and X3; and considering the deformations due to bending moment only, then, by "Virtual Work", the three conditions in equation (1) are given by:

$$\left.\begin{aligned} &\int \frac{M1\ M_o}{E\ I}\,dl + X_1 \int \frac{M1^2}{E\ I}\,dl + X_2 \int \frac{M1\ M2}{E\ I}\,dl + X_3 \int \frac{M1\ M3}{E\ I}\,dl = 0 \\ &\int \frac{M2\ M_o}{E\ I}\,dl + X_1 \int \frac{M2\ M1}{E\ I}\,dl + X_2 \int \frac{M2_2}{E\ I}\,dl + X_3 \int \frac{M2\ M3}{E\ I}\,dl = 0 \\ &\int \frac{M3\ M_o}{E\ I}\,dl + X_1 \int \frac{M3\ M1}{E\ I}\,dl + X_2 \int \frac{M3\ M2}{E\ I}\,dl + X_3 \int \frac{M3^2}{E\ I}\,dl = 0 \end{aligned}\right\} \quad . \quad (2)$$

When X1, X2, and X3 are determined, the final bending moment diagram M will be given by:

$$M = Mo + X1\ M1 + X2\ M2 + X3\ M3$$

Now, in this method, Mo, M1, M2 and M3 diagrams are first obtained by "Column Analogy". In many problems, especially those dealing with identical bays, some of these diagrams are not worked out separately as they are deduceable from each other.

In case of a three-vent frame as that shown in Fig. 6, the main system is chosen by cutting at the middle section of the frame. The redundant elements will be X1, X2, X3 which can be obtained from the three conditions that the vertical, horizontal

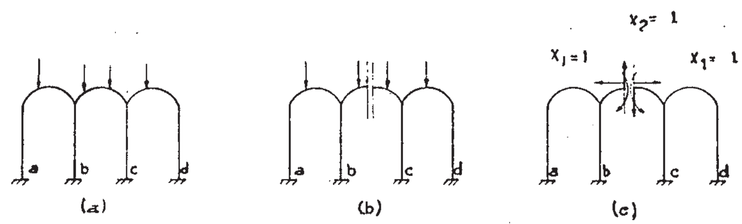


Fig. 6

and angular displacements are each equal to zero. Bending moment diagrams for X2 and X3 are deduceable from one operation. In addition, the $\int M1\ M2\ dl$, $\int M2\ M1\ dl$, $\int M2\ M3\ dl$ and $\int M3\ M2\ dl$ are each equal to zero, since the bending moment diagram due to $X2 = 1$ is of opposite sign on the two sides of the main system.

Conclusion:

The illustrated method furnishes a favourable solution for the two types of frames. The work involved is direct and fairly simple. In all cases, not more than three simultaneous equations are necessary.

For frames with straight roofs the operations involved become simpler and the work much reduced. The method applies with equal facility to frames of varying moment of inertia.

The method may also prove to be simpler than that of "Successive Approximation" or the "Virtual Work" alone.

PART 3

Experimental Investigation by Models of the Elastic Behaviour of Frames of Varying I

In analysing continuous frames of varying moment of inertia by moment distribution method, and by many other methods, such as "slope-deflection method", an axis of the frame is first decided upon.

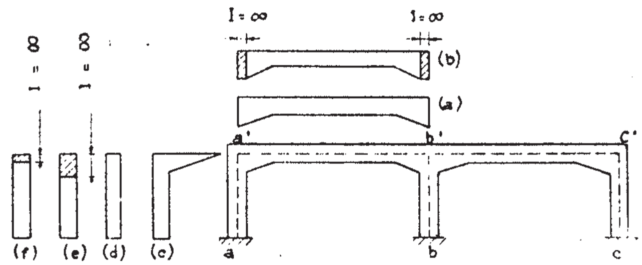


Fig. 7

Referring to Fig. 7, the axis is considered to be composed of the straight lines parallel to the straight edges of the various parts of the frame and passing through, or near, the centroid of the minimum section of each part. The frame is then considered to be formed of a number of members intersecting at points a', b' and c'.

No definite rule is followed in choosing these members, but several assumptions are in common use. Beams as a' b', are considered to extend to the axes of the adjoining columns as in Fig. 7-a, *OR* to have an infinite "I" over the parts shown in Fig. 7-b.

Columns as aa' are assumed to extend to the axis of the frame as haunched members as in Fig. 7-c ; *OR* considered to extend to the axis of the frame as prismatic members as in Fig. 7-d, *OR* to have an infinite moment of inertia "I" over the length between the intersectien of the beam and the face of the column as in Fig. 7-e, *OR* to have an infinite "I" over only a certain part of this length as in Fig. 7-f.

As for the method adopted for the location of the axis of the frame, no error is to be expected since it has been proved experimentally that the axis of a haunched member could be assumed as straight, and the fixed-end-moments, stiffness and carry-over factors, calculated according to this assumption agree very well with the experimental results.

As for the assumptions adopted in choosing the members, it is left to theory or experiment to ascertain whether any of these assumptions represents the actual behaviour of these intersecting members. It is worth mentioning that these assumptions lead to different results—as checked theoretically by the writers—especially those dealing with the ends of the columns. For this reason, a number of experiments has been conducted on models in order to establish an appropriate method for choosing the members.

Description of Experiments:

All models were prepared from "Perspex sheets $\frac{1''}{16}$ thick ; a semi-plastic material, which is a product of the I.C.I. It can be said that perspex proved to be an excellent material for model analysis.

Nine different models were tested, seven of these were two hinged symmetrical rectangular frames, all except one of varying "I", the other two models were three hinged symmetrical rectangular frames of varying "I". The ratio of the depth of the minimum section in a beam was taken as big as 1 : 3 ; in a column the part which varied in "I" extended to about a fifth of its length. The reason for taking these big proportions, which are about the maximum proportions that are taken in practice, was to have the difference—if any—in the analysis according to the assumptions marked and noticeable.

When testing a frame, it was placed horizontally on ball bearings $\frac{1''}{8}$ diameter, resting on a smooth table, with enough weights placed on the upper side of the frame to prevent buckling when given horizontal displacement. A horizontal displacement was given by a special device to the lower end of the frame (see photographs). The deflections of the various points chosen on the beam a' b' or' a' b' c' of the frame were measured by a deflectometer fitted to a prepared apparatus (see photograph).

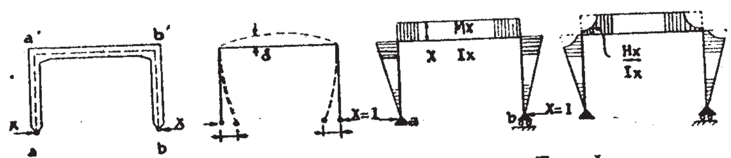


Fig. 8.—Single Intersection Joint ... Type I

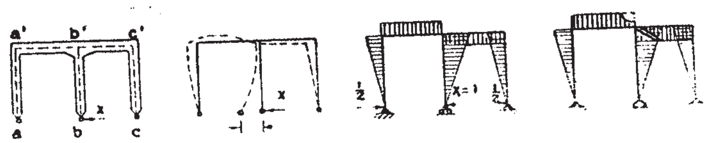


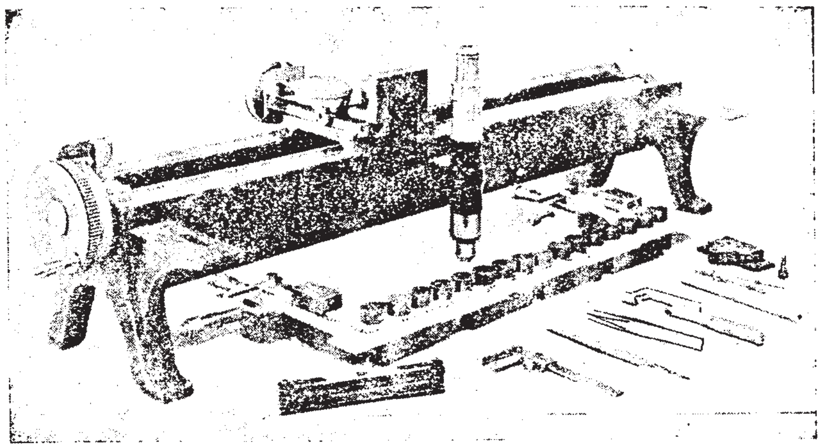
Fig. 9.—Double Intersection Joint ... Type II

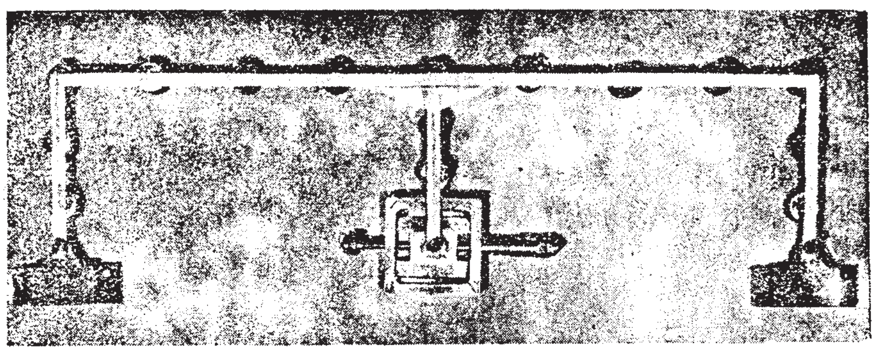
The influence line for the horizontal thrust "X" at a and b Fig. 8 and at b Fig. 9 were obtained experimentally and checked theoretically.

For type I, the frame was given two equal horizontal displacements $\delta 1/2$ at a and b, either inwards or outwards, while in type II, the frame was given at b, a horzontal displacement to the left side or to the right side with a and c kept hinged.

The influence line for "X" is then given (according to Maxwell's theorem of reciprocal deflection) by dividing each of the vertical deflections measured for points on a'b' or a'b'c' by δ_1. Thus, when the unit load is at point o Fig. 8., the value of $X = \frac{\delta_o}{\delta_1}$

The influence line is obtained theoretically by considering the case of loading $X = 1$; the corresponding bending moment diagram is to be drawn and modified according to the assumptions adopted for choosing the members of the frame mentioned before. This gives the "elastic weights". From the elastic weights on beam a'b' or a'b'c', the vertical deflections of its points are calculated as for a simple beam.





The horizontal movement δ_1 of the rollers du to $X = 1$ is given by $\int \frac{M^2}{EI} dl$. Again, by dividing the values of deflections computed for the points of a'b' or a'b'c' each by the corresponding value of δ_1, we obtain the theoretical influence line of X.

Conclusions :

From the experimental results compared with the theoretical study, the following were concluded :

1. The assumption which closely fitted the experimental values was the one in which beams are taken to the axis of the

frame, while columns are taken to extend to the axis of the beam as prismatic members having an infinite moment of inertia over a certain part "e" at its end, as in Fig. 7.-f.

For single intersection joints this part "e" was given by

$$e = 0{\cdot}5\ d + 0{\cdot}25\ h \ldots \quad \ldots \quad \ldots \quad (1)$$

nd for double intersection joints, this part was given by;

$$e = 0{\cdot}5\ d + 0{\cdot}75\ h \ldots \quad \ldots \quad \ldots \quad (2)$$

where d = depth of minimum section of beam and h = height from underside of minimum section of beam to point of junction of beam to column.

2. These simple rules gave good results ; the maximum difference between the experimental observations and the theoretical analysis being 5·2% only in a two hinged frame in which the haunch extended to about 1/3 the beam length.