THE VIERENDEEL GIRDER WITH EQUAL CHORD STIFFNESS

BY

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In the parallel chord Vierendeel girder it is the usual practice to give the same stiffness to the upper and lower chords. The same rule is often applied also to the polygonal chords, so that $\frac{I_u}{S_u} = \frac{I_L}{S_L}$ where I_u and I_L are the moments of inertia and S_u and S_L are the lengths of the upper and lower chords in each panel.

This special case of giving equal stiffness to the chords renders the calculation of the highly indeterminate Vierendeel girder relatively simple. The longitudinal deformations due to the axial forces are generally neglected. In this way, the vertical displacements of the upper and lower panel points will be the same. As a result, the upper and lower chord members will have equal deflections, so that, with equal chord stiffness, the moments along a vertical section through the girder will be the same, i.e. M' = M (Fig. 1)

Furthermore, if the Vierendeel girder is subject only to vertical panel loads, the horizontal thrusts in the two chords will be equal and opposite, i.e. H' = -H. Also, by taking two sections 2-2' and 3-3' at the ends of the panel 2-3, and considering the equilibrium of each chord separately, it will be easily seen that the value of H in the framed panel is constant (Fig. 1).

Whatever main system is adopted in the solution of this indeterminate structure, the redundant values, however, form together a system of internal forces and couples, which are in

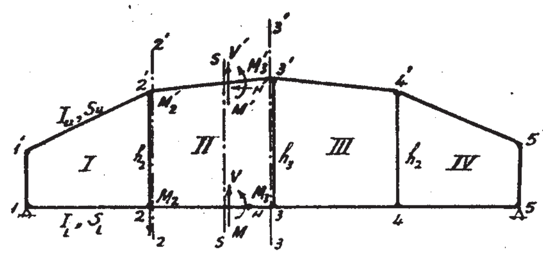
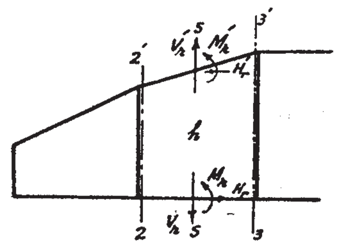


Fig. 1

equilibrium amongst themselves and which produce no reactions. Consequently, the corresponding moments, axial and shearing forces due to these redundant values along a vertical section s-s through the girder will be in equilibrium (Fig. 2). In other



$V'r = -Vr$
$H'r = -Hr$

Fig. 2

words $V'_r = -V_r$, $H'_r = -H_r$ and $M'_r = M_r = \frac{H_r . h}{2}$.

Referring to sections 2-2′ and 3-3′:

$$2\ M_{r2} = H_r .\ h_2 \text{ and } 2\ M_{r3} = H_r .\ h_3, \text{ and}$$

therefore $\frac{M_{r3}}{h_3} = \frac{M_{r2}}{h_2} = \frac{M_r}{h}$. Thus, the moments M in both chords due to the redundant values are equal and inversely proportional to the corresponding height of girder. It will be sufficient, therefore, to know a single redundant moment M_r in a certain panel in order to determine the rest of the M_r-diagrams of this panel.

Consequently, for a Vierendeel girder with equal chord stiffness and having n panels, there will be only n unknowns instead of 3n for the general type. The solution of the special type of Vierendeel girder is thus relatively simple.

A.—The Method of Elastic Couples

This method is suitable for the calculation of the special type of a Vierendeel girder with equal chord stiffness subject to indirect panel point loading. The idea involved therein is the introduction of the term "Elastic Couple", which is defined as the moment of two equal and opposite elastic weights. It is represented by a vector in the plane of the structure pointing in the direction of the forward motion of a right hand screw.

If now, the Vierendeel girder is referred to the main system shown in Fig. 3, the girder will act as a bow string girder with

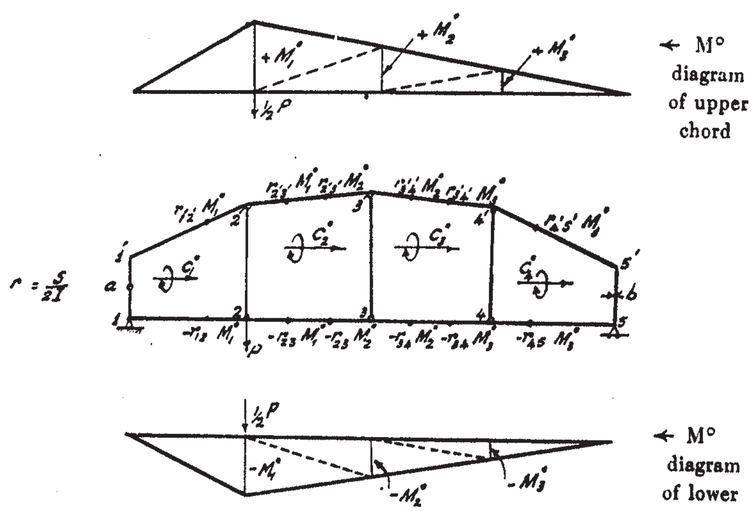


Fig. 3

stiff chords and hinged verticals. An additional hinge is further provided at "a", and a movable support is introduced at "b". The introduction of these hinges reduces the number of redundant values, but the system is still statically indeterminate.

Neglecting the longitudinal strain due to the axial forces, the deflections of the two chords will be equal. Remembering, also, that the chords have equal stiffness, the moments M° produced in the chords of the main system at every vertical section will be equal. However, with regard to the stresses produced in the chords themselves, these moments will have opposite signs. The corresponding elastic weights $\frac{M^\circ \, ds}{E \;\, I}$ will be equal and opposite. Consequently, the elastic weights of the upper and lower chords can be represented by normal vectors pointing respectively towards and away from the reader. Every two corresponding elastic weights thus form an elastic couple.

Referring to the bow string with stiff chords as a main system, Fig. 3, the moments M° due to the external loads will give the elastic couples C°. Since the verticals are hinged at both ends, these couples will arise from the chord members only. They fulfil the condition of equilibrium, but not the elastic conditions of the highly indeterminate Vierendeel girder.

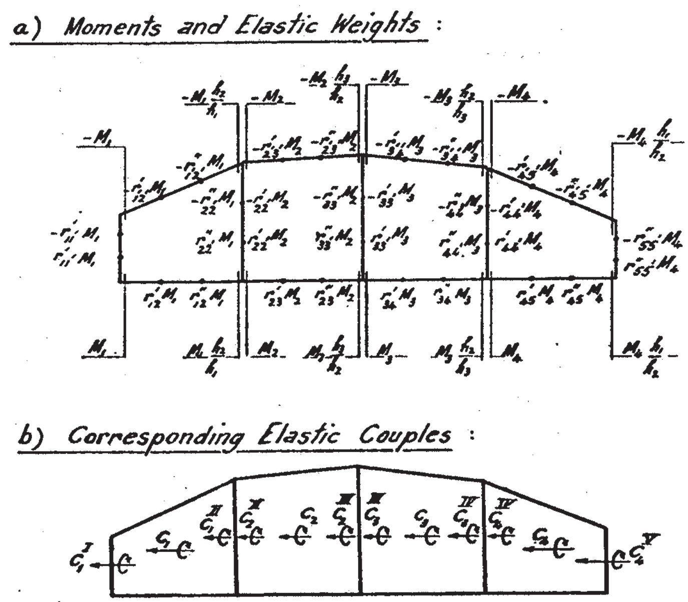


Fig. 4

Similarly, the moments M_r produced by the redundant values, Fig. 4, will give elastic couples C_r. This is clearly seen in the case of the equal chord moments. These moments, however, are transmitted to the verticals and will thus give pairs of equal and opposite elastic weights which in turn form elastic couples.

Remembering that the chord moments in each panel are proportional to the corresponding height, the elastic couples due to the chord moments in each panel will involve only one unknown. This applies also to the end verticals. The elastic couples due to the moments in the intermediate verticals, however, will contain two unknowns, namely, the chord moments of the two adjacent panels.

By adding the M_r-diagrams produced by the additional redundant values to the M^o-diagrams, the corresponding bending moment diagrams of the Vierendeel girder are obtained. Similar to the M^o- and M_r-diagrams, the resultant M-diagrams give equal values of moments in the upper and lower chords.

In order to fulfil the elastic conditions of the indeterminate Vierendeel girder, the elastic weights due to the resultant moments M in each closed panel must be in equilibrium. In other words, the sum of the resultant elastic couples in every panel must be zero.

Consequently, the elastic weights due to M_r should be in equilibrium with the elastic weights due to M^o, and the corresponding elastic couples should balance each other. This condition must be fulfilled in every panel, thus giving the key to the solution of the Vierendeel girder.

Starting at the left end panel, the elastic couples belonging to this panel are functions of the moments M_1 and M_2. If the elastic couple due to M^o is known, the equilibrium of the elastic couples supplies a relation between M_1 and M_2.

Going over to the next panel, it is now possible to express the elastie couples of the vertical 2-2′ as function of M_2 only. Consequently, the elastic couples in the second panel will be functions of M_2 and M_3. If the elastic couple due to M^o in this panel is known, the equilibrium of all elastic couples will give the relation between these two moments.

The process is continued from panel to panel until the right end panel is reached. Having found the relation between M_3 and M_4 from the equilibrium of the elastic couples in the preceding panel, the elastic weights and respective couples in this last panel will be functions of the moment M_4 only. This moment can be definitely found from the equilibrium of the elastic couples due to M and M° in this panel.

Having determined M_4, the solution is resumed going back from panel to panel and determining the unknown moments in every step. By the time the left end panel is reached, all redundant moments in the Vierendeel girder would have been obtained.

Numerical example:

Fig. 5a shows a Vierendeel girder with equal chord stiffness, hinged at one end and freely supported at the other. A single load $P = 1{\cdot}0$ ton is applied at joint 3.

The moment of inertia I of the straight upper chord is constant. On the other hand, the moment of inertia I_L of the polygonal lower chord is variable. It satisfies, however, the condition $I_L \cos \phi = I$, where ϕ = the inclination of the lower chord in each panel.

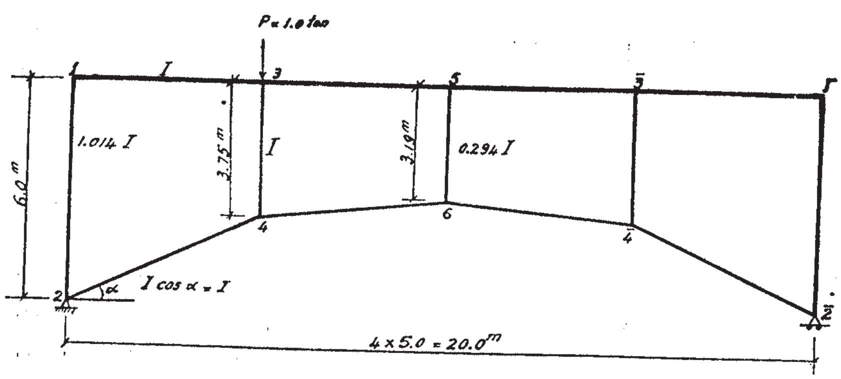


Fig. 5 a

The verticals 1-2, 3-4 and 5-6 have moments of inertia equal to 1·014 I, 1·00 I and 0·294 I respectively (¹).

The M° diagrams of the main system for upper and lower chords are drawn in Fig. 5b. As already explained, these two diagrams give equal and opposite values. The corresponding resultant elastic weights W° act at the third points of the different chord members. They are indicated at their points of application. Further, the end moments M which are produced by the redundant values are shown in Fig. 5c. The respective moments in the upper and lower chords are equal and opposite. In the same panel these moments are proportional to the height of the girder. The corresponding elastic weights W_r are concentrated in the third points of each member. They are written opposite to their points of application.

Starting at the left end panel, every pair of equal and opposite elastic weights is combined into an "Elastic Cou ", Figs. 5b and 5c. All couples containing the moment M_1 are

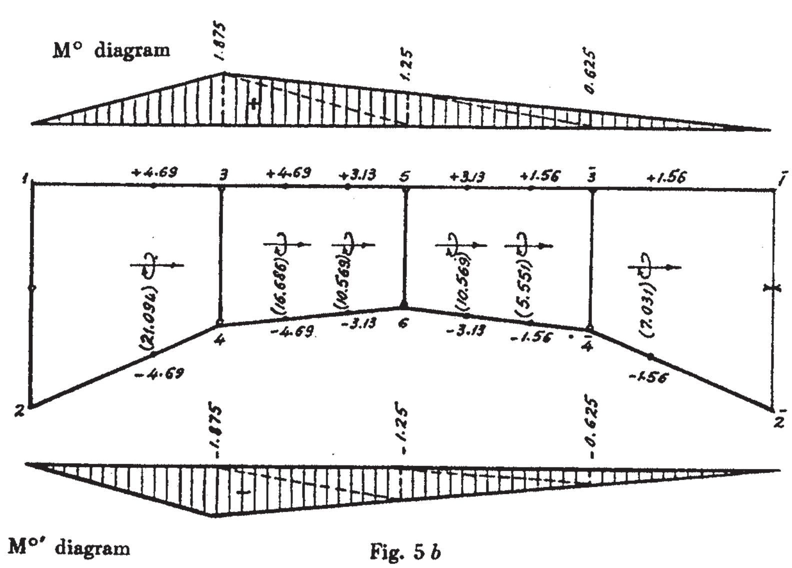


Fig. 5 *b*

(¹) This example is taken from Prof. L. C. Maugh's paper "Stresses and Deformations in Two-hinged Vierendeel Truss Arches", Proceedings of the Fifth International Congress of Applied Mechanics, 1938.

then added together, Fig. 6. The chords 1-3 and 2-4 supply two couples, while each of the verticals 1-2 and 3-4 gives one single couple in M_1. Further, there is another couple containing

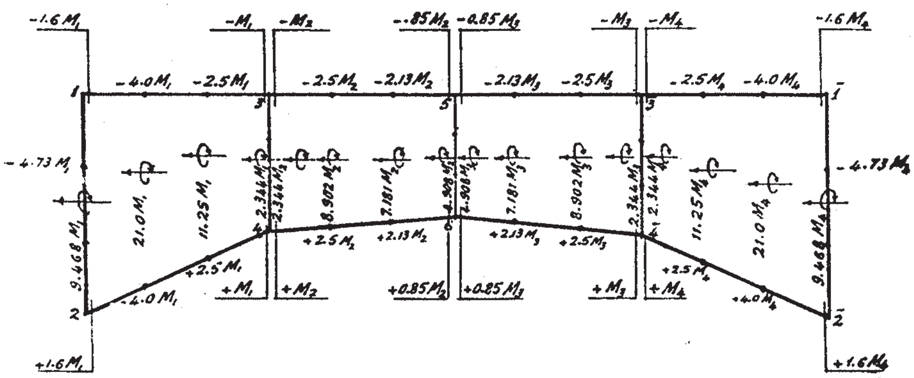
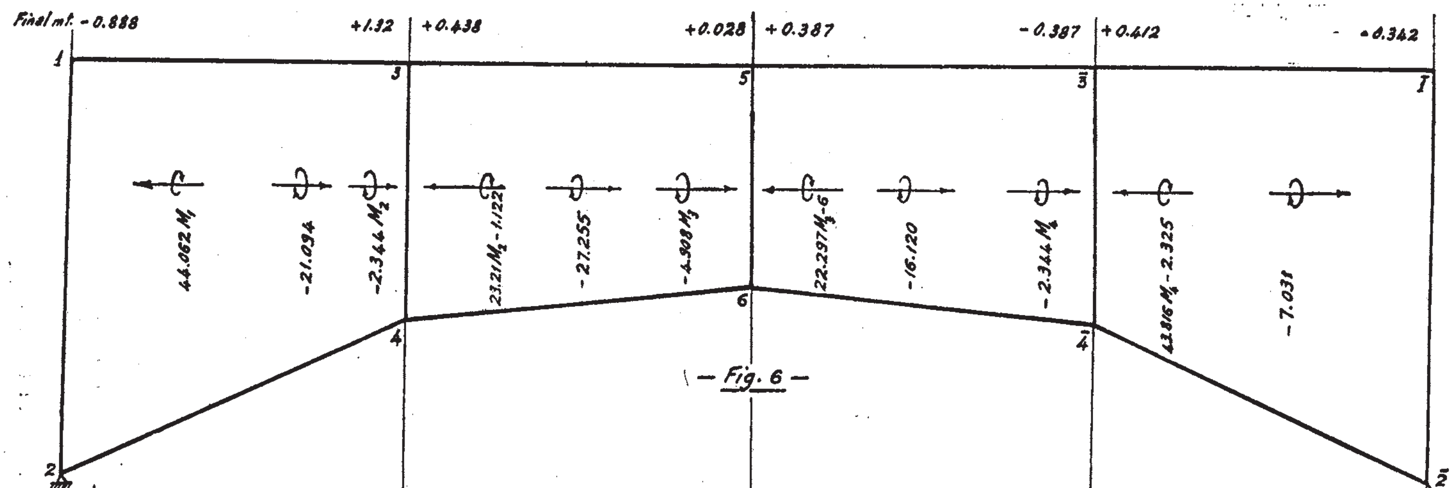


Fig. 5 c

the moment M_2 from the M_r-diagram of the vertical 3-4. There is also the elastic couple of the M° diagrams of the chord members 1-3 and 2-4. All six couples are now in equilibrium. This condition supplies the relation between M_1 and M_2 as shown under the first panel in Fig. 6.

Going over to the next panel, the vertical 3-4 supplies again two couples containing M_1 and M_2 respectively. The chord members 3-5 and 4-6, however, supply two couples containing M_2 only. Another two couples in M_2 and M_3 are supplied by the vertical 5-6. Further, there are two couples provided by the M° diagrams. On the whole, there are thus eight elastic couples in the second panel. However, the couple due to M_1 in the vertical 3-4 can be substituted by another couple in M_2 from the relation obtained before between M_1 and M_2.

Consequently, there will be five couples depending on M_2 which can be replaced by a single resultant couple. Also the two couples given by the M° diagrams can be added together. In this way, the second panel gives a couple in M_2 and a couple in M_3 which are in equilibrium with the resultant couple due to M°. This condition supplies the relation between M_2 and M_3 which is given under the second panel in Fig. 6.



— Fig. 6 —

M_1 couple $= 3.468 + 21.0 + 11.25 + 2.344 = 44.062$
M_2 couple $= -2.344\ M_2$
M_0 couple $= -21.094$

Total $= 0.0$

$\therefore 44.062\ M_1 - 2.344\ M_2 - 21.094 = 0$
$\therefore -2.344\ M_1 = -0.125\ M_2 - 1.122$

$\therefore M_1 = +0.555$

M_1 couple $= -2.344\ M_1 = -0.125\ M_2 - 1.122$
M_2 couple $= 2.344 + 8.902 + 7.181 + 4.908 = 23.335\ M_2$
M_1 couple $+ M_2$ couple $= 23.210\ M_2 - 1.122$
M_3 couple $= -4.908\ M_3$
M_0 couple $= -(16.686 + 10.569) = -27.255$

Total $= 0.0$

$\therefore 23.210\ M_2 - 4.908\ M_3 - 28.377 = 0$
$\therefore -4.908\ M_2 = -1.038\ M_3 - 6.0$

$\therefore M_2 = +1.437$

$-2.344\ M_1 = -0.125 \times 1.437 - 1.122$

M_2 couple $= -4.908\ M_2 = -1.038\ M_3 - 6.0$
M_3 couple $= 23.335\ M_3$
M_2 couple $+ M_3$ couple $= 22.297\ M_3 - 6.0$
M_4 couple $= -2.344\ M_4$
M' couple $= -(10.569 + 5.551) = -16.120$

Total $= 0.0$

$\therefore 22.297\ M_3 - 2.344\ M_4 - 21.120 = 0$
$\therefore -2.344\ M_3 = -0.246\ M_4 - 2.325$

$\therefore M_3 = +1.014$

$-4.908\ M_2 = -1.038 \times 1.014 - 6.0$

M_3 couple $= -2.344\ M_3 = -0.246\ M_4 - 2.325$
M_4 couple $= 44.062\ M_4$
M_3 couple $+ M_4$ couple $= 43.816\ M_4 - 2.325$
M' couple $= -7.031$

Total $= 0.0$

$\therefore 43.816\ M_4 - 9.356 = 0$

$\therefore M_4 = +0.213$

$-2.344\ M_3 = -0.246 \times 0.213 - 2.325$

A similar process is then carried out for the third panel giving the relation between M_3 and M_4. This relation is written under the third panel in Fig. 6.

Finally, the calculation is extended to the right-hand panel. All elastic couples are referred to the moment M_4. The equilibrium of all couples in this panel gives the numerical value of M_4 as shown.

Having found M_4, the end moment M_3 can be determined from the relation obtained before for the equilibrium of the elastic couples in the third panel. Similarly, M_2 is calculated from the relation between M_2 and M_3 obtained from the second panel. Finally, M_1 is found from the relation between M_1 and M_2 which has been derived from the equilibrium of the elastic couples in the first panel.

Needless to say, the process explained replaces the solution of the four elastic equations of the Vierendeel girder which are supplied by the equilibrium of the elastic couples in each panel. It is a method of solving these equations successively from panel to panel.

However, all elastic equations can be written down and solved algebraically in the ordinary way. They have the following form :

M_1	M_2	M_3	M_4	
a_1	a_2			$= a_0$
b_1	b_2	b_3		$= b_0$
	c_2	c_3	c_4	$= c_0$
		d_3	d_4	$= d_0$

The Vierendeel girder is thus referred to a system of 3 moment equations.

B.—The Method of Successive Approximations

(a) *The Panel Method* [1] :

If any panel abcd of the Vierendeel girder shown in Fig. 7a is separated by the sections 1-1 and 2-2, the forces and couples

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[1] Prof. L. C. Maugh, "Statically indeterminate Structures", 1947.

acting on the panel will be as shown in Fig. 7b. These forces and couples are of course in equilibrium with the external loads P_2 and P_3.

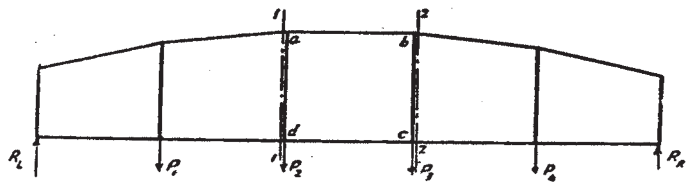

Fig. 7a

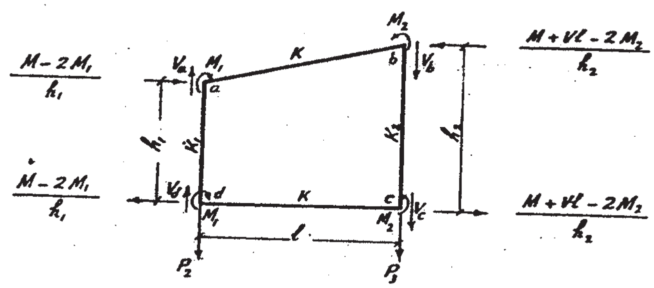

Fig. 7 b

To simplify the analysis, the force system in Fig. 7b is resolved into three equivalent force systems as shown in Fig. 7c, 7d and 7e. The forces given in Fig. 7c represent the action of the external forces, if the panel were hinged to the adjacent chord members. The force systems given by 7d and 7e are due

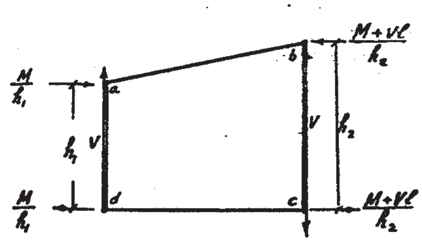

Fig. 7 c

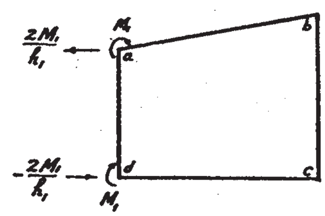

Fig. 7 d

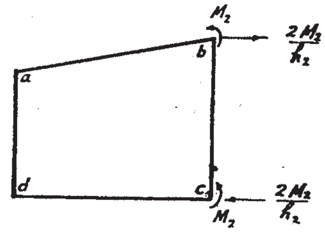

Fig. 7 e

to the moments M_1 and M_2 respectively. They represent the effect of continuity between the panels.

The following notations are used hereafter:

M = Bending moment at 1-1 due to external loads only

V = Shear in panel abcd

l = Panel length

h_1 and h_2 = Height of panel at 1-1 and 2-2

$K = \frac{I}{l}$ -value of chord members ab and cd

K_1 and K_2 = $\frac{I}{l}$ -values of members ad and cd

$$r = \frac{K}{K_1}, \quad s = \frac{K}{K_2}, \quad \alpha = \frac{h_2 - h_1}{h_1} \text{ and}$$

$$D = 6 + r + s + \alpha(2\alpha + \alpha s + 2s + 6)$$

The moments in the panel due to the force system of Fig. 7c are essential for the stability of the structure. For this reason, they have been designated "Primary Moments".

$$M'_{ab} = M'_{dc} = \frac{\alpha M - Vl}{2D}[3 + s + \alpha(2 + s)] \text{ and}$$

$$M'_{ba} = M'_{cd} = \frac{\alpha M - Vl}{2D}(3 + r + \alpha)$$

They can be obtained directly as end moments by the slope deflection equations.

The force systems shown in Figs. 7d and 7e represent the effect of the internal moments in the adjacent panels, and, therefore, exist as a result of continuity of the panels. The moments M'' and M''' produced by these two force systems respectively are not necessary for the structure stability in the panel abcd. They will be defined as "Secondary Moments". The magnitude of these secondary moments can be computed without difficulty by the following equations :

$$M''_{ab} = M''_{dc} = +\frac{r}{D} M_1$$

$$M''_{ba} = M''_{cd} = -\frac{r(1+\alpha)}{D} M_1$$

$$M'''_{ab} = M'''_{dc} = -\frac{s(1+\alpha)}{D} M_2$$

$$M'''_{ba} = M'''_{cd} = +\frac{s(1+\alpha)^2}{D} M_2$$

Evidently the constants $\frac{r}{D}$, $\frac{r(1+\alpha)}{D}$, $\frac{s(1+\alpha)}{D}$ and $\frac{s(1+\alpha)^2}{D}$ are correction factors that can be computed for each panel. The primary moments M' in the adjacent panels can be used for the first approximation of M_1 and M_2, and the corrections can be computed as for any method of successive approximations.

Numerical example :

In order to illustrate the "Panel Method" explained above, the same numerical example given before is solved hereafter.

Primary Moments M'

$$M'_{ab} = M'_{a'b'} = \frac{-0{\cdot}75 \times 5}{2 \times 5{\cdot}505} [3+0{\cdot}751-0{\cdot}375\ (2+0{\cdot}761)] = -0{\cdot}925$$

$$M'_{ba} = M'_{b'a'} = \frac{-0{\cdot}75 \times 5}{2 \times 5{\cdot}505} (3+1{\cdot}184-0{\cdot}375) = -1{\cdot}298$$

$$M'_{bc} = M'_{b'c'} = \frac{-0{\cdot}149 \times 3{\cdot}75+0{\cdot}25 \times 5}{2 \times 7{\cdot}484} [3+2{\cdot}17-0{\cdot}149 \times (2+2{\cdot}17)] = +0{\cdot}210$$

$$M'_{cb} = M'_{c'b'} = \frac{-0{\cdot}149 \times 3{\cdot}75+0{\cdot}25 \times 5}{2 \times 7{\cdot}484} (3+0{\cdot}764-0{\cdot}149) = +0{\cdot}167$$

$$M'_{cd} = M'_{c'd'} = \frac{0{\cdot}176 \times 2{\cdot}5+0{\cdot}25 \times 5}{2 \times 10{\cdot}346} [3+0{\cdot}764+0{\cdot}176 \times (2+0{\cdot}764)] = +0{\cdot}347$$

$$M'_{dc} = M'_{d'c'} = \frac{0{\cdot}176 \times 2{\cdot}5+0{\cdot}25 \times 5}{2 \times 10{\cdot}346} (3+2{\cdot}170+0{\cdot}176) = +0{\cdot}437$$

$$M'_{dc} = M'_{d'c'} = \frac{0{\cdot}6 \times 1{\cdot}25 + 0{\cdot}25 \times 5}{2 \times 14{\cdot}102} \; [3 + 1{\cdot}184 + 0{\cdot}6 \times (2 + 1{\cdot}184)] = +0{\cdot}432$$

$$M'_{ed} = M'_{e'd'} = \frac{0{\cdot}6 \times 1{\cdot}25 + 0{\cdot}25 \times 5}{2 \times 14{\cdot}102} \; (3 + 0{\cdot}751 + 0{\cdot}6) = +0{\cdot}308$$

The corrections due to continuity are obtained from the recorded correction factors. The order in which the corrections are carried out is indicated by arrows. For example, the corrections in the second panel for the primary moment 1·298 in the first panel are :

$$M''_{bc} = -\,(0{\cdot}1022)\;(-1{\cdot}298) \qquad = +\,0{\cdot}132$$

$$M''_{cb} = +\,(0{\cdot}0869)\;(-1{\cdot}298) \qquad = -\,0{\cdot}113$$

Further, the corrections in the third panel for the corrected moment $(+\,0{\cdot}167 - 0{\cdot}113 = 0{\cdot}054)$ are :

$$M''_{cd} = -\,(0{\cdot}21)\;(0{\cdot}054) \qquad = -\,0{\cdot}011$$

$$M''_{dc} = +\,(0{\cdot}2466)\;(0{\cdot}054) \qquad = +\,0{\cdot}013$$

The process is continued in this way until no more corrections are needed. The process converges after four successive corrections as seen in Fig. 8.

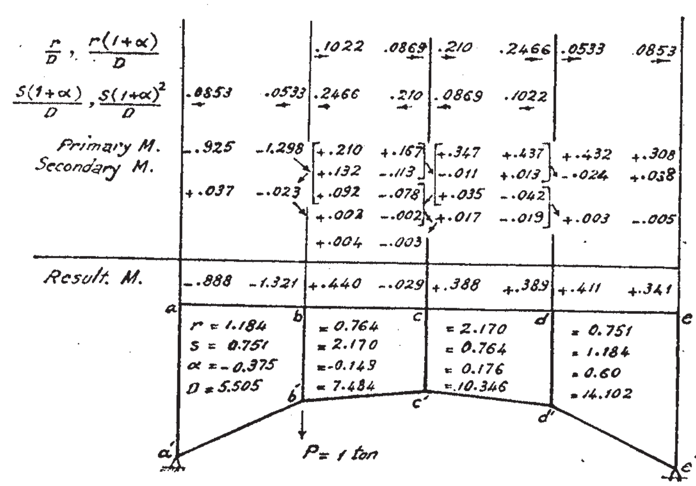

Fig. 8

Modification of the Panel Method:

The panel method explained in the previous paragraph assumes hinges just outside the panel in question. The moments of the Vierendeel girder at these points are not minimum. On the contrary, the bending moment diagrams for panel point loading show straight lines from panel to panel, so that the maximum bending moments occur just where the hinges are introduced. For this reason, several corrections are needed until the process finally converges. Furthermore, the first corrections at least will be, at certain points, nearly of the same magnitude as the primary moments.

The idea involved in the modification, which will be proposed, is to introduce the hinges as far as possible at the points of contraflexure, which are expected in the Vierendeel girder. Referring to the bending moment diagrams of such a system, it is easily seen that the points of zero moments coincide to a fair degree of approximation with the middle points of the panels. If, now, two halves of the adjacent panels are added to each panel, and the method of successive approximations is applied to such division, the number of corrections needed will be reduced, and the secondary moments will be relatively small in comparison with the primary moments.

Fig. 9 shows one of the proposed divisions and the force system acting upon it. Such a division is obtained by taking sections 1-1 and 2-2 through the mid-points of the two adjacent panels. Similar to the panel method explained before, this force system is divided into the primary system shown in Fig. 9a, and the two secondary systems given in Figs. 9b and 9c respectively. Here again, the force system in Fig. 9a represents the forces acting upon the division, if the four ends were really hinged. The two other force systems in Figs. 9b and 9c represent the effect of continuity.

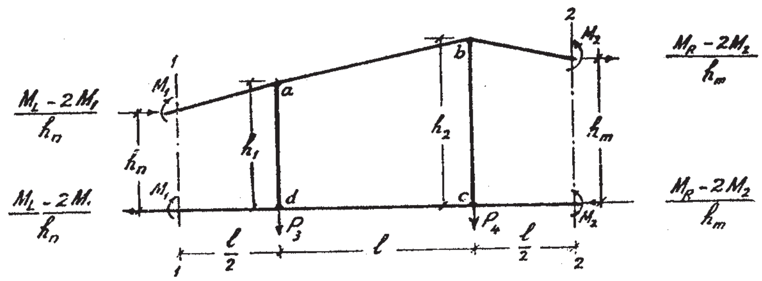

Fig. 9

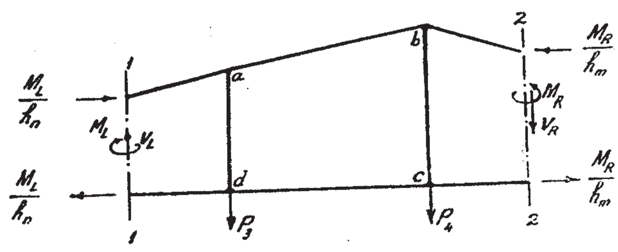

Fig. 9 a

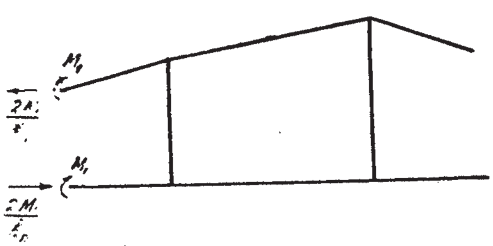

Fig. 9 b

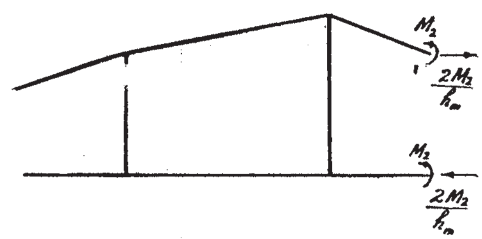

Fig. 9 c

In working out the end moments due to the primary force system, the panel method as explained before can be applied. Fig. 10a shows the force system on the panel abcd, due to the primary force system shown in Fig. 9a. The loads acting at the mid-points of the adjacent panels are simply shifted parallel to themselves to the corners of the panel abcd, introducing four couples as indicated. This system is now treated as if it were built up of primary and secondary systems. Using the same coefficients as in the previous panel method, the corresponding moments are :

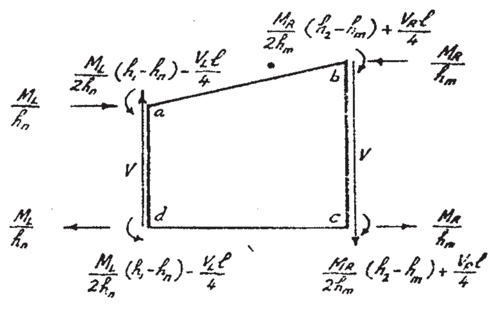

Fig. 10 *a*

$$M'_{ab}=M'_{dc}=\frac{\alpha(M_L+0{\cdot}5V_L l)-Vl}{2D}(3+s+\alpha(2-s)-\frac{r}{D}\Big(\frac{M_L}{2h_n}\times$$

$$(H_1-h_n)-\frac{V_L l}{4}\Big)-\frac{s(1+\alpha)}{D}\Big(\frac{M_R}{2h_m}(h_2-h_m)+\frac{V_R l}{4}\Big)$$

and $$M'_{ba}=M'_{cd}=\frac{\alpha(M_L+0{\cdot}5V_L l)-Vl}{2D}(3+r+\alpha)+\frac{r(1+\alpha}{D}\times$$

$$\Big(\frac{M_L}{2h_n}(h_1-h_n)-\frac{V_L l}{4}\Big)+\frac{s(1+\alpha)^2}{D}\Big(\frac{M_R}{2h_m}(h_2-h_m)-\frac{V_R l}{4}\Big)$$

where M_L = B.M. on section 1-1 due to external loads only

V_L = shear at section 1-1,

M_R = B.M. on section 2-2 due to external loads only

V_R = shear at section 2-2

h_n and h_m = heights of girder at sections 1-1 and 2-2 respectively

h_1 and h_2 = heights of girder at ends of panel.

These moments are due to the force system shown in Fig. 9a, and are thus the new primary moments in the modified panel method.

The force systems shown in Figs. 9b and 9c can be treated in a similar way. The forces and couples acting at the mid-points of the adjacent panels are shifted to the corners of the panel abcd, introducing additional couples. The corresponding force systems are shown in Figs. 10b and 10c respectively. They give directly the following secondary moments :

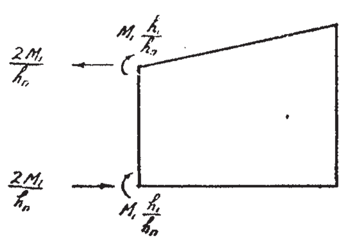

Fig. 10 *b*

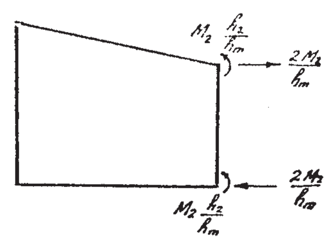

Fig. 10 *c*

$$M'_{ab} = M'_{dc} = + \frac{r}{D}\,\frac{h_1}{h}\,M_1$$

$$M'_{ba} = M'_{cd} = - \frac{r\,(1+\alpha)}{D}\,\frac{h_1}{h_n}\,M_1$$

$$M''_{ab} = M''_{dc} = - \frac{s\,(1+\alpha)}{D}\,\frac{h_2}{h_m}\,M_2$$

$$M'''_{ba} = M'''_{cd} = + \frac{s\,(1+\alpha)^2}{D}\,\frac{h_2}{h_m}\,M_2$$

As already mentioned, these secondary moments are very small compared with the primary moments. They can even be neglected. In other words, the primary moments obtained by the proposed modification give, to a good degree of approximation, the bending moment diagrams of the Vierendeel girder. There is hardly any need of further correction.

In order to prove this statement, the same numerical example worked out before is solved hereafter by the "Modified Panel Method".

Numerical example (Fig. 11):

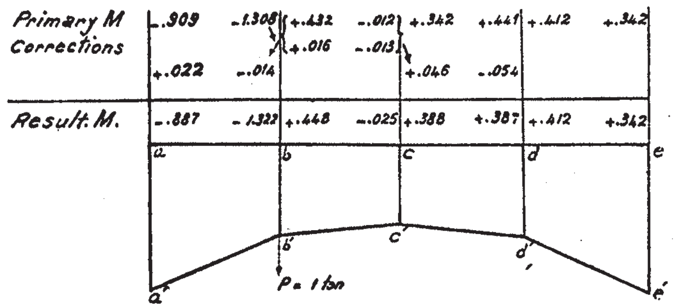

Fig. 11

The primary moments M' are:

$$M'_{ab} = M'_{a'b'} = \frac{-0\ 75 \times 5}{2 \times 5{\cdot}505} \left[3+0{\cdot}751-0{\cdot}375\ \ (2+0{\cdot}751)\right]$$
$$-0\ 0853 \left(\frac{3{\cdot}125}{2 \times 3{\cdot}47}\ (3{\cdot}75-3{\cdot}47) - \frac{0{\cdot}25 \times 5}{4} \right) = -0{\cdot}909$$

$$M'_{ba} = M'_{b'a'} = \frac{-0{\cdot}75 \times 5}{2 \times 5{\cdot}505}\ (3+1{\cdot}184-0{\cdot}375)\ +0{\cdot}0533\ (-0{\cdot}1864) = -1{\cdot}308$$

$$M'_{bc} = M'_{b'c'} = \frac{-0{\cdot}149\ (1{\cdot}875+0{\cdot}5 \times 0{\cdot}75 \times 5)+0{\cdot}25 \times 5}{2 \times 7{\cdot}484} \times$$
$$[3+2{\cdot}17 - 0{\cdot}149\ (2+2{\cdot}17)] - 0{\cdot}1022 \times \left(\frac{1{\cdot}875}{2 \times 4{\cdot}875}\ (3{\cdot}75-4{\cdot}875) - \frac{0{\cdot}75 \times 5}{4} \right)$$
$$-0{\cdot}2466 \left(\frac{1{\cdot}875}{2 \times 3{\cdot}47}\ (3{\cdot}19-3{\cdot}47) - \frac{0{\cdot}25 \times 5}{4} \right) = +0{\cdot}432$$

$$M'_{cb} = M'_{c'b'} = 0{\cdot}047\ (3+0{\cdot}764-0{\cdot}149)+0{\cdot}0869\ (-1{\cdot}154) +0{\cdot}21\ (-0{\cdot}389) = -0{\cdot}012$$

$$M'_{cd} = M'_{c'd'} = \frac{0{\cdot}176\ \ (3{\cdot}125-0{\cdot}5 \times 0{\cdot}25 \times 5)\ \ +0{\cdot}25 \times 5}{2 \times 10{\cdot}346} \times$$
$$[3+0{\cdot}764+0{\cdot}176\ \ (2+0{\cdot}764)] - 0{\cdot}210 \times$$

$$\left(\frac{3{\cdot}125\ (3{\cdot}19-3{\cdot}47)}{2\times3{\cdot}47}+\frac{0{\cdot}25\times5}{4}\right)-0{\cdot}0869\times$$
$$\left(\frac{0{\cdot}625(3{\cdot}75-4{\cdot}875)}{2\times4{\cdot}875}-\frac{0{\cdot}25\times5}{4}\right)=+0{\cdot}342$$

$$M'_{dc}=M'_{d'c'}=+0{\cdot}082\ (3+2{\cdot}17+0{\cdot}176)+0{\cdot}2466\ (0{\cdot}187)$$
$$+0{\cdot}1022\ (-0{\cdot}389)=+0{\cdot}441$$

$$M'_{de}=M'_{d'e'}=\frac{0{\cdot}6\ (1{\cdot}875-0{\cdot}5\times0{\cdot}25\times5)\ +0{\cdot}25\times5}{2\times14{\cdot}102}\times$$
$$[3+1{\cdot}184+0{\cdot}6\ (2+1{\cdot}184)]-0{\cdot}0533\ \times$$
$$\left(\frac{1{\cdot}875\ (3{\cdot}75-3{\cdot}47)}{2\times3{\cdot}47}+\frac{0{\cdot}25\times5}{4}\right)=+0{\cdot}412$$

$$M'_{ed}=M'_{e'd'}=0{\cdot}071\ (3+0{\cdot}751+0{\cdot}6)+0\ 0853\ (0{\cdot}389)$$
$$=+0{\cdot}342$$

The respective corrections are:

$$M''_{bc}=-0{\cdot}1022\times\frac{3{\cdot}75}{4{\cdot}875}\left(\frac{-1{\cdot}308+0{\cdot}909}{2}\right)$$
$$=-0{\cdot}1022\ (-0{\cdot}153)=+0{\cdot}016$$

$$M''_{cb}=+0{\cdot}0869\ (-0{\cdot}153)=-0{\cdot}013$$

$$M''_{ba}=-0{\cdot}0533\times\frac{3{\cdot}74}{3{\cdot}47}\left(\frac{0{\cdot}448+0{\cdot}025}{2}\right)$$
$$=-0{\cdot}0533\ (+0{\cdot}256)=-0{\cdot}014$$

$$M''_{ab}=+0{\cdot}0869\ (+0{\cdot}256)=+0{\cdot}022$$

$$M''_{cd}\ \ -0{\cdot}210\times\frac{3{\cdot}19}{3{\cdot}47}\left(\frac{-0{\cdot}025-0{\cdot}448}{2}\right)$$
$$=-0{\cdot}210\ (-0{\cdot}217)=+0{\cdot}046$$

$$M''_{dc}=+0{\cdot}2466\ (-0{\cdot}217)=-0{\cdot}054$$

$$M''_{de}=-0{\cdot}0533\times\frac{3{\cdot}75}{3{\cdot}47}\left(\frac{+0{\cdot}387-0{\cdot}388}{2}\right)\cong0{\cdot}000$$
$$M''_{ed}=\qquad\simeq0{\cdot}000$$

The whole process is shown in Fig. 11.

·e accurate results are obtained after one single correction. Even ..is correction is not absolutely necessary. In the ordinary panel method, it is reminded, four successive corrections were neede to arrive at the final results. This justifies the proposed modification.

C.—The Vierendeel Girder with Constant Stiffness Ratio

The method of "Elastic Couples" as well as the "Panel Method" have been established for the special type of a Vierendeel girder with equal chord stiffness, subject to panel point loading. The general case of variable chord stiffness and direct loading is obviously more difficult.

In the following, a trial is made to use the panel method for the approximate solution of a Vierendeel girder with constant stiffness ratio. The two chords have different stiffness, yet a constant ratio between the upper and lower chord stiffness is here maintained.

The idea involved is to replace the system by a Vierendeel girder with equal chord stiffness of a value equal to the mathematical average of the real two values of chord stiffness. The results obtained by the panel method show that up to about 30% difference in the upper and lower chord stiffness, there will be fair approximations of the real values.

Table 1 gives the accurate end moments for panel point loading when the upper chord stiffness is increased 30%, while keeping the stiffness of the lower chord and vertical members unchanged. The end moments calculated for an average constant stiffness of 1·15 are shown in Table 2.

A study of the results shows that, by increasing the stiffness of the upper chord, bigger end moments are produced in these members. The moments obtained by assuming a constant average stiffness lie between the corresponding values of upper and lower chords.

Table 3 shows the percentage error in the maximum ordinates, if the values of Table 2 are used as approximations for those in Table 1. It contains also the percentage error, if the load covers the whole span, *i.e.* in the area of the influence lines.

TABLE 1

Moments	Unit load at		
	3	5	3
M_{13}	—0·894	—0·673	—0·347
M_{31}	1·437	0·867	0·424
M_{35}	0·553	—0·531	—0·391
M_{53}	0·065	1·094	0·409
M_{24}	0·842	0·643	0·332
M_{42}	—1·228	—0·810	—0·401
M_{46}	—0·383	0·516	0·375
M_{64}	—0·042	—0·892	—0·378

TABLE 2

Moments	Unit load at		
	3	5	3
$M_{13} = -M_{24}$	—0·869	—0·658	—0·340
$M_{31} = -M_{42}$	1·332	0·839	0·413
$M_{35} = -M_{46}$	0·467	—0·525	—0·384
$M_{53} = -M_{64}$	0·052	0·990	0·392

TABLE 3

Moments	Area of I.L.		% error	Max. ordinate		% error
	Exact	Approx.		Exact	Approx.	
M_{13}	—9·570	—9·330	2·5	—0·894	—0·869	2·88
M_{31}	13·640	12·920	5·57	1·437	1·332	7·88
M_{35}	—1·845	—2·210	16·52	0·553	0·467	18·41
M_{53}	7·845	7·205	8·8	1·094	0·990	10·61

It is clear that errors become less if the load covers a longer part of the span.

With a smaller difference in the chord stiffness than 30%, the proposed method gives better approximations. The error involved in the maximum ordinate with a difference of 20°/₀ for example is 5·65°/₀ only, and with 10°/₀ difference, the same error is further reduced to 2·87°/₀.

A study of the results shows, further, that the proposed approximate moments lie between the real values of the upper and lower chords. Consequently, the sum of the real and approximate moments respectively will be almost the same. This means that the normal forces produced by assuming a constant average stiffness will be almost identical with those found in the case of unequal chord stiffness.

Remembering now, that the maximum stresses are affected by the axial forces as well as by the bending moments, it becomes evident that ths maximum stresses, calculated by the proposed method of constant average stiffness, will show a better degree of approximation than that for the bending moment alone.

Further, the big errors indicated in Table 3 refer to the absolute maximum differences in ordinates and areas of influence lines for the end moments. These values, however, are not always governing. The percentage error is less in the case of the bigger moments which control the design.

In short, the proposed method of assuming an average chord stiffness gives, with differences of 10 and 20°/₀ in the chord stiffness, good results. Taking all factors into consideration, this method is also useful as a practical approximation up to an appreciable difference of 30 and even 50°/₀ in the chord stiffness.